UR-1422
ER-40685-871
hep-th/9505093

Supersymmetric Two Boson Equation, Its Reductions and the Nonstandard Supersymmetric KP Hierarchy

J. C. Brunelli

and

Ashok Das

Department of Physics and Astronomy
University of Rochester
Rochester, NY 14627, USA

Abstract

In this paper, we review various properties of the supersymmetric Two Boson (sTB) system. We discuss the equation and its nonstandard Lax representation. We construct the local conserved charges as well as the Hamiltoniam structures of the system. We show how this system leads to various other known supersymmetric integrable models under appropriate field redefinition. We discuss the sTB and the supersymmetric nonlinear Schrödinger (sNLS) equations as constrained, nonstandard supersymmetric Kadomtsev-Petviashvili (sKP) systems and point out that the nonstandard sKP systems naturally unify all the KP and mKP flows while leading to a new integrable supersymmetrization of the KP equation. We construct the nonlocal conserved charges associated with the sTB system and show that the algebra of charges corresponds to a graded, cubic algebra. We also point out that the sTB system has a hidden supersymmetry making it an $N = 2$ extended supersymmetric system.

1. Introduction

Integrable models have been studied in the past from many different points of view [1-3]. These models have a very rich structure and as such deserve to be studied in their own right. Recently, however, these models have become quite relevant in the study of strings through the matrix models [4]. This has generated additional interest, particularly, in the supersymmetric integrable models since they are the ones which are likely to be relevant in the study of superstrings [5,6].

While the bosonic integrable models have been studied quite extensively, not much is known, in general, about the supersymmetric integrable models. The most widely studied supersymmetric integrable system is the supersymmetric KdV (sKdV) system [7,8] and has many interesting properties. Another super integrable model which has a very rich structure and which, in fact, leads to the sKdV system upon reduction is the supersymmetric KP equation of Manin-Radul [7]. While some of its properties are known, a lot remains to be known about the properties of this highly nontrivial system.

There is another supersymmetric integrable system that also has an equally rich structure and is the subject of discussion of this paper. It is the supersymmetric generalization of what is known as the Two Boson system [9]. This system, known as the supersymmetric Two Boson (sTB) system [10] is rich in the sense that it gives rise to many other supersymmetric integrable systems under appropriate field redefinitions or reductions – much like the sKP system. Yet, its structure is simple enough to determine many of its properties explicitly. In this paper, we review all the known properties of this system.

Our paper is organized as follows. In section 2, we discuss the Two Boson system [9] and discuss briefly various known properties of this system. In section 3, we construct the supersymmetric generalization of this system (sTB) [10] and discuss its Lax representation as well as the local conserved charges associated with it. In section 4, we generalize the three Hamiltonian structures of the bosonic system to the superspace and construct the recursion operator which relates different Hamiltonians of the system. We point out that this is truly a bi-Hamiltonian super integrable system (in the sense that there are two local, compatible Hamiltonian structures) – in fact, the only one that we are aware of. In section 5, we discuss how the supersymmetric nonlinear Schrödinger equation (sNLS), the sKdV

equation and the smKdV equations can be obtained from this system. The Hamiltonian structures of these systems can also be obtained from those of the sTB system and we indicate how this is carried out [11]. The Lax operator for the sTB system gives rise to a scalar Lax operator for the sNLS equation which shows that the sNLS system can be thought of as a constrained, nonstandard supersymmetric KP system (sKP) and this is discussed in section 6. This constrained, nonstandard sKP system unifies all the KP and mKP flows into a single equation and leads to a new supersymmetric form of the KP equation that is integrable [12]. In section 7, we construct the conserved nonlocal charges associated with the sTB system and show that their algebra is an interesting, cubic graded algebra. This also shows that the sTB equation has a second supersymmetry which is not manifest making it an $N = 2$ extended supersymmetric system [13]. We present our conclusions in section 8. In particular, we clarify the relation between some recently proposed equations with that of ours [14-17]. We compile some of the necessary technical formulae in the Appendix.

2. Two Boson Hierarchy

The dispersive generalization of the long water wave equation [9,18,19] in a narrow channel has the form

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= (2h + u^2 - \alpha u')' \\
\frac{\partial h}{\partial t} &= (2uh + \alpha h')'
\end{aligned}
\tag{2.1}
$$

where $u(x,t)$ and $h(x,t)$ can be thought of as the horizontal velocity and the height (vertical displacement from the mean depth), respectively, of the free surface (α is an arbitrary parameter) and a prime denotes a derivative with respect to x. This system of equations is integrable [19] and has a tri-Hamiltonian structure [9]. It has a nonstandard Lax representation and reduces to various known integrable systems with appropriate identification [9]. For example, with the identification

$$
\begin{aligned}
\alpha &= 1 \\
u &= -q'/q \\
h &= q\bar{q}
\end{aligned}
\tag{2.2}
$$

we get the nonlinear Schrödinger equation [20-23], with $\alpha = -1$, $h = 0$, (2.1) gives the Burgers' equation while for $\alpha = 0$ we obtain Benney's equation which is the standard long wave equation. The KdV and the mKdV equations can also be obtained from the hierarchy associated with this integrable equation. Thus, (2.1) in different limits, describes the long wave equation (Benney's equation, KdV equation) as well as the short wave equation (the nonlinear Schrödinger equation) [24]. This equation, therefore, has a very rich structure much like the KP equation in two dimensions. In fact, as we will see, it has a lot in common with the KP equation which has been studied extensively in the literature.

From now on we will choose $\alpha = 1$ and we will make the identification $u = J_0$ and $h = J_1$, which is used in the Two Boson formulation of this system [20,25]. The equations in (2.1) then take the form

$$\begin{aligned} \frac{\partial J_0}{\partial t} &= (2J_1 + J_0^2 - J_0')' \\ \frac{\partial J_1}{\partial t} &= (2J_0 J_1 + J_1')' \end{aligned} \tag{2.3}$$

In this form, the equations are commonly referred to as the Two Boson equation (TB) and this is the form of the equations that we will use in our discussions. It is easy to see that the variables of the system can be assigned the following canonical dimensions.

$$[x] = -1 \qquad [t] = -2 \qquad [J_0] = 1 \qquad [J_1] = 2 \tag{2.4}$$

The system of equations (2.3) can be written in the form of a Lax equation [9,25]

$$\frac{\partial L}{\partial t} = \left[L, \left(L^2\right)_{\geq 1}\right] \tag{2.5}$$

where the Lax operator has the form

$$L = \partial - J_0 + \partial^{-1} J_1 \tag{2.6}$$

and $()_{\geq 1}$ refers to the differential part of a pseudo-differential operator. This is conventionally called a nonstandard representation of the Lax equation. We note that canonical dimension of the Lax operator is given by

$$[L] = 1 \tag{2.7}$$

The conserved quantities of the system (which are in involution) are given by

$$H_n = \text{ Tr } L^n = \int dx \text{ Res } L^n \qquad\qquad n = 1, 2, 3, \ldots \tag{2.8}$$

where "Res" stands for the coefficient of the ∂^{-1} term in the pseudo-differential operator and the first few Hamiltonians have the explicit forms

$$H_1 = \int dx\ J_1 \qquad H_2 = \int dx\ J_0 J_1 \qquad H_3 = \int dx\ \left(J_1^2 - J_0' J_1 + J_1 J_0^2\right) \tag{2.9}$$

Given a Lax equation, one can obtain the Hamiltonian structures associated with the dynamical equations through the Gelfand-Dikii brackets in a straightforward manner at least for standard Lax representations [3,26]. However, the present system corresponds to a nonstandard Lax representation [27] and consequently, the standard definitions of the Gelfand-Dikii brackets need to be generalized in this case. There are two possible, but equivalent, generalizations which lead to the correct Hamiltonian structures of the system [25]. First, for the Lax operator

$$L = \partial + \overline{J}_0 + \partial^{-1}\overline{J}_1 \qquad\qquad \left(\overline{J}_0 = -J_0 \quad \overline{J}_1 = J_1\right) \tag{2.10}$$

the standard definition of a dual

$$Q = q_0 + q_1 \partial^{-1} \tag{2.11}$$

leads to a linear functional of $\left(\overline{J}_0, \overline{J}_1\right)$ of the form

$$F_Q(L) = \text{ Tr } LQ = \sum_{i=0}^{1} \int dx\ q_{1-i}(x) \overline{J}_i(x) \tag{2.12}$$

We can modify the definition of the dual as

$$\overline{Q} = Q + q_{-1}\partial \tag{2.13}$$

with the constraint (The structure of the Lax equations leads to such constraints [25].)

$$q_{-1} J_1 = q_1 \tag{2.14}$$

which would lead to the Hamiltonian structures from the standard definition of Gelfand-Dikii brackets as

$$\begin{aligned}
\left\{F_{\overline{Q}}(L), F_{\overline{V}}(L)\right\}_1 &= \text{Tr } L[\overline{Q}, \overline{V}] \\
\left\{F_{\overline{Q}}(L), F_{\overline{V}}(L)\right\}_2 &= \text{Tr } ((L(\overline{V}L)_+ - (L\overline{V})_+ L)\overline{Q})
\end{aligned} \tag{2.15}$$

Equivalently, we can keep the definition of the dual in (2.11) and modify the definition of the Gelfand-Dikii brackets as

$$\begin{aligned}
\left\{F_Q(L), F_V(L)\right\}_1 &= \text{Tr } L[Q, V] \\
\left\{F_Q(L), F_V(L)\right\}_2 &= \text{Tr } ((L(VL)_+ - (LV)_+ L)Q) \\
&\quad + \int dx \left[\left(\int^x \text{Res } [Q, L]\right) \text{Res } [V, L]\right. \\
&\quad \left. + \text{ Res } [Q, L] \text{ Res } \left(\partial^{-1} LV\right) - \text{ Res } [V, L] \text{ Res } \left(\partial^{-1} LQ\right)\right]
\end{aligned} \tag{2.16}$$

Either way, the first two Hamiltonian structures of the system can be derived from (2.15) or (2.16). The first structure turns out to be simple

$$\begin{pmatrix} \left\{J_0(x), J_0(y)\right\}_1 & \left\{J_0(x), J_1(y)\right\}_1 \\ \left\{J_1(x), J_0(y)\right\}_1 & \left\{J_1(x), J_1(y)\right\}_1 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \delta(x-y) = \mathcal{D}_1 \delta(x-y) \tag{2.17}$$

where $\partial \equiv \frac{\partial}{\partial x}$. Similarly, the second structure is easily obtained to be

$$\begin{pmatrix} \left\{J_0(x), J_0(y)\right\}_2 & \left\{J_0(x), J_1(y)\right\}_2 \\ \left\{J_1(x), J_0(y)\right\}_2 & \left\{J_1(x), J_1(y)\right\}_2 \end{pmatrix} = \begin{pmatrix} 2\partial & \partial J_0 - \partial^2 \\ J_0 \partial + \partial^2 & J_1 \partial + \partial J_1 \end{pmatrix} \delta(x-y) = \mathcal{D}_2 \delta(x-y) \tag{2.18}$$

Note that if we introduce the recursion operator

$$R = \mathcal{D}_2 \mathcal{D}_1^{-1} \tag{2.19}$$

we can obtain a third Hamiltonian structure for the system as

$$\mathcal{D}_3 = R\,\mathcal{D}_2 \tag{2.20}$$

whose explicit form is complicated and not very interesting, except for the fact that it is local. But we note that eqs. (2.17), (2.18) and (2.20) define the three Hamiltonian structures of the Two Boson equation (hierarchy) as

$$\partial_t \begin{pmatrix} J_0 \\ \\ J_1 \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta J_0} \\ \\ \frac{\delta H_3}{\delta J_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta J_0} \\ \\ \frac{\delta H_2}{\delta J_1} \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta J_0} \\ \\ \frac{\delta H_1}{\delta J_1} \end{pmatrix} \tag{2.21}$$

making it a tri-Hamiltonian system [9]. It is worth emphasizing here that integrable systems are in general multi-Hamiltonian [1-3], but what is special about the Two Boson equation is that the three distinct Hamiltonian structures we have just derived are local. It is also easy to check that the conserved charges of the system are recursively related by the recursion operator as

$$\begin{pmatrix} \frac{\delta H_{n+1}}{\delta J_0} \\ \\ \frac{\delta H_{n+1}}{\delta J_1} \end{pmatrix} = R^\dagger \begin{pmatrix} \frac{\delta H_n}{\delta J_0} \\ \\ \frac{\delta H_n}{\delta J_1} \end{pmatrix} \tag{2.22}$$

We also note here for completeness that, if we define

$$\begin{aligned} J(x) &= J_0(x) \\ T(x) &= J_1(x) - \frac{1}{2}\, J_0'(x) \end{aligned} \tag{2.23}$$

then, the second Hamiltonian structure in (2.18), in these variables, has the form [25]

$$\begin{aligned} \{J(x), J(y)\}_2 &= 2\partial_x \delta(x-y) \\ \{T(x), J(y)\}_2 &= J(x)\partial_x \delta(x-y) \\ \{T(x), T(y)\}_2 &= (T(x) + T(y))\partial_x \delta(x-y) + \frac{1}{2}\, \partial_x^3 \delta(x-y) \end{aligned} \tag{2.24}$$

which is a Virasoro-Kac-Moody algebra for a $U(1)$ current [28,29] (also known as an affine algebra [30]). The second Hamiltonian structure, in (2.18) or (2.24), as we will see in section 8, corresponds to the bosonic limit of the $N = 2$ twisted superconformal algebra [31].

As we have already noted earlier, with the field identifications [20-23]

$$\begin{aligned} J_0 &= -\, \frac{q'}{q} = -(\ln q)' \\ J_1 &= \bar{q} q \end{aligned} \tag{2.25}$$

we obtain from (2.3), the nonlinear Schrödinger equation

$$\begin{aligned}\frac{\partial q}{\partial t} =& -\left(q'' + 2(\bar{q}q)q\right)\\ \frac{\partial \bar{q}}{\partial t} =& \,\bar{q}'' + 2(\bar{q}q)\bar{q}\end{aligned} \tag{2.26}$$

The Lax operator in (2.6), with the identification in (2.25), factorizes and has the form

$$\begin{aligned}L =& \partial + \frac{q'}{q} + \partial^{-1}\bar{q}q\\ =& G\widetilde{L}G^{-1}\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}G =& q^{-1}\\ \widetilde{L} =& \partial + q\partial^{-1}\bar{q}\end{aligned} \tag{2.28}$$

We say that the two Lax operators, L and $\widetilde{L}$, are related through a gauge transformation [20,23,32]. The new Lax operator, $\widetilde{L}$, allows us to write the nonlinear Schrödinger equation in the standard Lax representation

$$\frac{\partial \widetilde{L}}{\partial t} = \left[\widetilde{L}, (\widetilde{L}^2)_+\right] \tag{2.29}$$

providing a scalar Lax operator for the nonlinear Schrödinger equation which puts it in the same footing as the other integrable equations such as the KdV equation. The Lax operator, $\widetilde{L}$, in (2.28) can also be written as

$$\begin{aligned}\widetilde{L} =& \partial + q\bar{q}\,\partial^{-1} - q\bar{q}'\partial^{-2} + q\bar{q}''\partial^{-3} + \cdots\\ =& \partial + \sum_{n=0}^{\infty} u_n\partial^{-n-1}\end{aligned} \tag{2.30}$$

with

$$u_n = (-1)^n q\bar{q}^{(n)} \tag{2.31}$$

where $f^{(n)}$ is the nth derivative of f with respect to x. The form of $\widetilde{L}$ in (2.30) is the same as that of a KP system with coefficient functions constrained by (2.31). This allows us to think of the nonlinear Schrödinger equation as a constrained KP system [20,22,27].

Some observations are in order here for the discussion of the supersymmetric generalization of this system which we are going to take up in the coming sections. Let us note that given $\widetilde{L}$, we can define its formal adjoint [33]

$$\mathcal{L} = \widetilde{L}^* = -(\partial + \bar{q}\partial^{-1}q) \tag{2.32}$$

and the standard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\left(\mathcal{L}^2\right)_+, \mathcal{L}\right] \tag{2.33}$$

also gives the nonlinear Schrödinger equation. Furthermore, with the identification

$$\bar{q} = q \tag{2.34}$$

the standard Lax equation

$$\frac{\partial \widetilde{L}}{\partial t} = \left[\widetilde{L}, (\widetilde{L}^3)_+\right] \tag{2.35}$$

leads to the mKdV equation,

$$\frac{\partial q}{\partial t} = -(q''' + 6q^2q') \tag{2.36}$$

The Lax operator, $\mathcal{L}$, in (2.32) with the same identification as in (2.34), becomes

$$\mathcal{L} = -\widetilde{L} \tag{2.37}$$

and also gives the mKdV equation from

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\left(\mathcal{L}^3\right)_+, \mathcal{L}\right] \tag{2.38}$$

This shows how the mKdV equation can be embedded into the nonlinear Schrödinger equation and from our discussion above, it is clear that both the Lax operator and its formal adjoint yield equivalent results. We will see later that this equivalence no longer holds in the supersymmetric generalization of this system.

3. Supersymmetric Two Boson Hierarchy

The easiest way to derive the supersymmetric generalization of the Two Boson equation is to go to the superspace [10]. Let $z = (x, \theta)$ define the coordinates of the superspace with θ representing the Grassmann coordinate and

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \tag{3.1}$$

representing the supercovariant derivative. We see from (3.1) that $D^2 = \partial$, and this leads to

$$[\theta] = -\frac{1}{2} \tag{3.2}$$

Let us now introduce two fermionic superfields

$$\begin{aligned} \Phi_0 &= \psi_0 + \theta J_0 \\ \Phi_1 &= \psi_1 + \theta J_1 \end{aligned} \tag{3.3}$$

which have the following canonical dimensions

$$\begin{aligned} [\Phi_0] &= [\psi_0] = \frac{1}{2} \\ [\Phi_1] &= [\psi_1] = \frac{3}{2} \end{aligned} \tag{3.4}$$

This would correspond to a simple extension of the variables of our bosonic system to an $N = 1$ supersymmetric case.

With the superfields in (3.3) as our basic variables, we can now write the most general, local dynamical equations in superspace which are consistent with all the canonical dimensions and which reduce to (2.3) in the bosonic limit. These are easily checked to be

$$\begin{aligned} \frac{\partial \Phi_0}{\partial t} =& -(D^4\Phi_0) + 2(D\Phi_0)(D^2\Phi_0) + 2(D^2\Phi_1) \\ & + a_1 D(\Phi_0(D^2\Phi_0)) + a_2 D(\Phi_0\Phi_1) \\ \frac{\partial \Phi_1}{\partial t} =& (D^4\Phi_1) + b_1 D((D^2\Phi_1)\Phi_0) + 2(D^2\Phi_1)(D\Phi_0) - b_2 D(\Phi_1(D^2\Phi_0)) \\ & + 2(D\Phi_1)(D^2\Phi_0) + b_3\Phi_1\Phi_0(D^2\Phi_0) + b_4 D(\Phi_1\Phi_0)(D\Phi_0) \\ & + b_5 D(\Phi_0(D^4\Phi_0)) + b_6 D(\Phi_0(D^2\Phi_0))(D\Phi_0) \end{aligned} \tag{3.5}$$

where a_1, a_2, b_1, b_2, b_3, b_4, b_5 and b_6 are arbitrary parameters. An integrable system is likely to have a Lax representation and, therefore, we look for a Lax representation for the supersymmetric equations (3.5). It is easy to see that a consistent Lax equation can be defined if we choose, as the Lax operator, the pseudo super-differential operator

$$L = D^2 + \alpha(D\Phi_0) + \beta D^{-1}\Phi_1 \tag{3.6}$$

where α, β are arbitrary parameters and $D^{-1} = \partial^{-1}D$ is the inverse of the covariant derivative operator. (Note that for $\alpha = -1$ and $\beta = 1$ (3.6) reduces to (2.6) in the bosonic limit.) The nonstandard Lax equation

$$\frac{\partial L}{\partial t} = \left[L, (L^2)_{\geq 1}\right] \tag{3.7}$$

in this case, gives

$$\begin{aligned}\frac{\partial \Phi_0}{\partial t} &= -(D^4\Phi_0) - 2\alpha(D\Phi_0)(D^2\Phi_0) - \frac{2\beta}{\alpha}\,(D^2\Phi_1)\\ \frac{\partial \Phi_1}{\partial t} &= (D^4\Phi_1) - 2\alpha D^2((D\Phi_0)\Phi_1)\end{aligned} \tag{3.8}$$

Comparing with (3.5), we conclude that both the equations are compatible only if $\alpha = -1$ and $\beta = 1$ so that

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1 \tag{3.9}$$

and the most general supersymmetric extension of the dynamical equations in (2.3) which is integrable is given by [10]

$$\begin{aligned}\frac{\partial \Phi_0}{\partial t} &= -(D^4\Phi_0) + (D(D\Phi_0)^2) + 2(D^2\Phi_1)\\ \frac{\partial \Phi_1}{\partial t} &= (D^4\Phi_1) + 2(D^2((D\Phi_0)\Phi_1))\end{aligned} \tag{3.10}$$

In components, (3.10) gives rise to the following system of interacting equations

$$\begin{aligned}\frac{\partial J_0}{\partial t} &= (2J_1 + J_0^2 - J_0')'\\ \frac{\partial \psi_0}{\partial t} &= 2\psi_1' + 2\psi_0' J_0 - \psi_0''\\ \frac{\partial J_1}{\partial t} &= (2J_0J_1 + J_1' + 2\psi_0'\psi_1)'\\ \frac{\partial \psi_1}{\partial t} &= (2\psi_1 J_0 + \psi_1')'\end{aligned} \tag{3.11}$$

And it is straightforward to check that it is invariant under the supersymmetry transformations

$$\begin{aligned}\delta J_0 &= \epsilon\psi_0'\\ \delta J_1 &= \epsilon\psi_1'\\ \delta\psi_0 &= \epsilon J_0\\ \delta\psi_1 &= \epsilon J_1\end{aligned} \tag{3.12}$$

where ϵ is a constant Grassmann parameter. This, therefore, appears to be the most general $N = 1$ supersymmetric extension of the Two Boson equation which is integrable. However, as we will see later, there are hidden symmetries in this system [13]. In particular, among other things, we will see that there is a second supersymmetry [13,15,16] associated with this system.

Equation (3.10) or (3.11) define an integrable system and therefore, there are an infinite number of local conserved charges which can be obtained in the standard manner to be

$$Q_n = \mathrm{sTr}\, L^n = \int dz\, \mathrm{sRes}\, L^n \qquad n = 1, 2, \ldots \tag{3.13}$$

where "sRes" stands for the super residue which is defined to be the coefficient of the D^{-1} term in the pseudo super-differential operator with D^{-1} at the right. The constancy of these charges under the flow of (3.10) can, of course, be directly checked, but it also follows easily from the structure of the Lax equation in (3.7). The first few charges have the explicit form

$$\begin{aligned}
Q_1 =& -\int dz\, \Phi_1 \\
Q_2 =& 2\int dz\, (D\Phi_0)\Phi_1 \\
Q_3 =& 3\int dz\, \Big[(D^3\Phi_0) - (D\Phi_1) - (D\Phi_0)^2\Big]\Phi_1 \\
Q_4 =& 2\int dz\, \Big[2(D^5\Phi_0) + 2(D\Phi_0)^3 + 6(D\Phi_0)(D\Phi_1) - 3\left(D^2(D\Phi_0)^2\right)\Big]\Phi_1
\end{aligned} \tag{3.14}$$

We observe that $[Q_n] = n$, they are bosonic and are invariant under the supersymmetry transformations given by (3.12). In fact, since the supersymmetry transformations (3.12) can be thought of as arising from a translation of the Grassmann coordinate in the superspace, and since these charges are given as superspace integrals of local functions, supersymmetry of these charges is manifest.

We also note here that associated with the supersymmetric Two Boson equation is a hierarchy of supersymmetric integrable equations given by

$$\frac{\partial L}{\partial t_n} = [L, (L^n)_{\geq 1}] \tag{3.15}$$

with L given in eq. (3.9). All the equations in (3.15) share the same conserved charges and define the supersymmetric Two Boson (sTB) hierarchy. In the next section, we will derive the Hamiltonian structures associated with this hierarchy.

4. Hamiltonian Structures for the sTB

As we have noted earlier, given a Lax equation, we can obtain the Hamiltonian structures associated with a dynamical system from the Gelfand-Dikii brackets [3,26]. The sTB hierarchy, as we have seen, has a nonstandard Lax representation. While the generalization of the Gelfand-Dikii brackets for the bosonic nonstandard Lax equations was obtained in ref. [25], the extension of these brackets to superspace is technically more difficult and not yet understood. Therefore, we will construct the brackets by supersymmetrizing the bosonic Hamiltonian structures (2.17), (2.18) and (2.20) in a direct way [10,13].

Defining the Hamiltonians of the sTB system as

$$H_n = \frac{(-1)^{n+1}}{n} Q_n \tag{4.1}$$

we note that the sTB equation (3.10) can be written as a Hamiltonian system with three Hamiltonian structures of the form [10,13]

$$\partial_t \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta \Phi_0} \\ \\ \frac{\delta H_3}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta \Phi_0} \\ \\ \frac{\delta H_2}{\delta \Phi_1} \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta \Phi_0} \\ \\ \frac{\delta H_1}{\delta \Phi_1} \end{pmatrix} \tag{4.2}$$

where the first structure has the local form

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \tag{4.3}$$

This can also be written as a matrix in the component space as

$$\mathcal{D}_1 = \begin{pmatrix} 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.4}$$

and yields the following nonvanishing Poisson brackets in components

$$\begin{aligned} \{\psi_0(x), \psi_1(y)\}_1 &= -\,\delta(x-y) \\ \{J_0(x), J_1(y)\}_1 &= \delta'(x-y) \end{aligned} \tag{4.5}$$

The second Hamiltonian structure of the system is given by [13]

$$\mathcal{D}_2 = \begin{pmatrix} -2D - 2D^{-1}\Phi_1 D^{-1} + D^{-1}(D^2\Phi_0)D^{-1} & D^3 - D(D\Phi_0) + D^{-1}\Phi_1 D \\ \\ -D^3 - (D\Phi_0)D - D\Phi_1 D^{-1} & -\Phi_1 D^2 - D^2\Phi_1 \end{pmatrix} \tag{4.6}$$

and can also be written as a matrix in the component space as

$$\mathcal{D}_2 = \begin{pmatrix} 2\partial & \partial J_0 - \partial^2 & (\partial\psi_0)\partial^{-1} - 2\psi_1\partial^{-1} & \psi_1 \\ J_0\partial + \partial^2 & \partial J_1 + J_1\partial & -(\partial\psi_0) - \partial\psi_1\partial^{-1} & \partial\psi_1 + \psi_1\partial \\ \partial^{-1}(\partial\psi_0) - 2\partial^{-1}\psi_1 & (\partial\psi_0) + \partial^{-1}\psi_1\partial & \partial^{-1}((\partial J_0) - 2J_1)\partial^{-1} - 2 & \partial^{-1}J_1 - J_0 + \partial \\ -\psi_1 & \partial\psi_1 + \psi_1\partial & -J_1\partial^{-1} - J_0 - \partial & 0 \end{pmatrix} \tag{4.7}$$

which gives the nonvanishing Poisson brackets in the components of the form

$$\begin{aligned}
\{\psi_0(x), \psi_0(y)\}_2 =& \Big(\partial^{-1}J_0'\big(\partial^{-1}\delta(x-y)\big)\Big) - 2\Big(\partial^{-1}J_1\big(\partial^{-1}\delta(x-y)\big)\Big) - 2\delta(x-y) \\
\{\psi_0(x), J_0(y)\}_2 =& \Big(\partial^{-1}\psi_0'\delta(x-y)\Big) - 2\Big(\partial^{-1}\psi_1\delta(x-y)\Big) \\
\{J_0(x), J_0(y)\}_2 =& 2\delta'(x-y) \\
\{\psi_0(x), \psi_1(y)\}_2 =& \Big(\partial^{-1}J_1\delta(x-y)\Big) - J_0\delta(x-y) + \delta'(x-y) \\
\{\psi_0(x), J_1(y)\}_2 =& \psi_0'\delta(x-y) + \Big(\partial^{-1}\psi_1\delta'(x-y)\Big) \\
\{J_0(x), \psi_1(y)\}_2 =& \psi_1\delta(x-y) \\
\{J_0(x), J_1(y)\}_2 =& (J_0\delta(x-y))' - \delta''(x-y) \\
\{\psi_1(x), J_1(y)\}_2 =& 2\psi_1\delta'(x-y) + \psi_1'\delta(x-y) \\
\{J_1(x), J_1(y)\}_2 =& J_1'\delta(x-y) + 2J_1\delta'(x-y)
\end{aligned} \tag{4.8}$$

Introducing the recursion operator

$$R = \mathcal{D}_2\mathcal{D}_1^{-1} \tag{4.9}$$

the third Hamiltonian structure can be written as

$$\mathcal{D}_3 = R\,\mathcal{D}_2 \tag{4.10}$$

whose explicit form is complicated and uninteresting.

While the skew-symmetry character of the Hamiltonian structures in (4.3), (4.6) and (4.10) are easy to see, proof of the Jacobi identity for these structures, at least, for $\mathcal{D}_2$ and $\mathcal{D}_3$ is not at all obvious. Neither is the bi-Hamiltonian character of this system (which follows from the compatibility of the Hamiltonian structures or the fact that a linear superposition of the structures is also a Hamiltonian structure). We will now sketch how these can be checked for the Hamiltonian structures of our system using prolongation methods described in [34] and generalized to the supersymmetric systems in [35]. Let us define a two component column matrix of bosonic superfields as

$$\vec{\Omega} = \begin{pmatrix} \Omega_0 \\ \\ \Omega_1 \end{pmatrix} \tag{4.11}$$

and construct the bivector $\Theta_{\mathcal{D}}$ associated with any Hamiltonian structure $\mathcal{D}$ as

$$\Theta_{\mathcal{D}} = \frac{1}{2}\sum_{\alpha,\beta}\int dz\ ((\mathcal{D})_{\alpha\beta}\Omega_\beta)\wedge\Omega_\alpha \qquad \alpha,\beta = 0,1 \tag{4.12}$$

Then, a necessary and sufficient condition for $\mathcal{D}$ to define a Hamiltonian structure is that the prolongation of this bivector must vanish (We refer interested readers to refs. [34] and [35] for details.)

$$\mathbf{pr}\,\vec{v}_{\mathcal{D}\vec{\Omega}}(\Theta_{\mathcal{D}}) = 0 \tag{4.13}$$

For $\mathcal{D}_2$, the second Hamiltonian structure of sTB, it is straightforward to show that

$$\begin{aligned}\mathbf{pr}\,\vec{v}_{\mathcal{D}_2\vec{\Omega}}(\Phi_0) = &-2(D\Omega_0) - 2\left(D^{-1}(\Phi_1(D^{-1}\Omega_0))\right) + \left(D^{-1}(D^2\Omega_0)(D^{-1}\Omega_0)\right)\\ &+ (D^3\Omega_1) - (D^2\Phi_0)\Omega_1 - (D\Phi_0)(D\Omega_1) + \left(D^{-1}(\Phi_1(D\Omega_1))\right)\end{aligned} \tag{4.14a}$$

$$\begin{aligned}\mathbf{pr}\,\vec{v}_{\mathcal{D}_2\vec{\Omega}}(\Phi_1) = &-(D^3\Omega_0) - (D\Phi_0)(D\Omega_0) - (D\Phi_1)(D^{-1}\Omega_0) + \Phi_1\Omega_0\\ &- 2\Phi_1(D^2\Omega_1) - (D^2\Phi_1)\Omega_1\end{aligned} \tag{4.14b}$$

Using this, the prolongation of the bivector (4.12) for the second structure is easily seen to vanish

$$\mathbf{pr}\,\vec{v}_{\mathcal{D}_2\vec{\Omega}}(\Theta_{\mathcal{D}_2}) = 0 \tag{4.15}$$

implying that $\mathcal{D}_2$ satisfies the Jacobi identity. Also, since $\mathcal{D}_1$ is field independent, it follows directly that

$$\mathbf{pr}\,\vec{v}_{\mathcal{D}_1\vec{\Omega}}(\Theta_{\mathcal{D}_1}) = 0 \tag{4.16}$$

It can be shown similarly that $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible, in that if we define

$$\mathcal{D} = \mathcal{D}_2 + \alpha\mathcal{D}_1 \tag{4.17}$$

where α is an arbitrary constant, then

$$\mathbf{pr}\,\vec{v}_{\mathcal{D}\vec{\Omega}}(\Theta_{\mathcal{D}}) = \mathbf{pr}\,\vec{v}_{\mathcal{D}\vec{\Omega}}(\Theta_{\mathcal{D}_2}) + \alpha\mathbf{pr}\,\vec{v}_{\mathcal{D}\vec{\Omega}}(\Theta_{\mathcal{D}_1}) = 0 \tag{4.18}$$

showing that any arbitrary linear superposition of the first and the second Hamiltonian structures is also a Hamiltonian structure. In a similar manner, it can be shown that the third structure, $\mathcal{D}_3$, is also Hamiltonian and that the supersymmetric Two Boson equation is a tri-Hamiltonian system much like its bosonic counterpart.

Several comments are in order here. First, the two higher structures, $\mathcal{D}_2$ and $\mathcal{D}_3$, are not local unlike the bosonic equation. As we will show in sec. 8, the second Hamiltonian structure really becomes local and coincides with the $N = 2$ twisted superconformal algebra [31] in a different basis (in variables which are linearly related). This, therefore, represents the only supersymmetric system that we know of which has two local Hamiltonian structures and in this sense represents a truly bi-Hamiltonian system. We would also like to emphasize here that even though the Hamiltonian structures obtained in ref. [10] (through a naive supersymmetrization of the bosonic Hamiltonian structures) give the correct dynamical equations as well as the correct recursion relations between the first few conserved charges, they fail to satisfy Jacobi identity. The structures in (4.3), (4.6) and (4.10) represent the true Hamiltonian structures of the system.

Finally, we note here that the recursion operator defined in (4.9) from the Hamiltonian structures, relates the conserved Hamiltonians of the system recursively as

$$\begin{pmatrix} \frac{\delta H_{n+1}}{\delta\Phi_0} \\ \\ \frac{\delta H_{n+1}}{\delta\Phi_1} \end{pmatrix} = R^{\dagger} \begin{pmatrix} \frac{\delta H_n}{\delta\Phi_0} \\ \\ \frac{\delta H_n}{\delta\Phi_1} \end{pmatrix} \tag{4.19}$$

This is sufficient to show that the conserved charges are in involution (This can also be seen directly from the Lax equation.) leading to the integrability of the system. We note here, for later use, the explicit form of this operator

$$R^{\dagger} = \begin{pmatrix} D^2 - D^{-1}(D^2\Phi_0) + (D\Phi_0) + \Phi_1 D^{-1} & 2(D\Phi_1) - 2\Phi_1 D - D^{-1}(D^2\Phi_1) \\ \\ 2 + D^{-2}\Phi_1 D^{-1} - D^{-2}(D^2\Phi_0)D^{-1} & -D^2 - D^{-2}(D\Phi_1) + (D\Phi_0) + D^{-1}\Phi_1 \end{pmatrix} \tag{4.20}$$

In the next section, we will show how this supersymmetric system reduces to various other known supersymmetric integrable systems, much like its bosonic counterpart.

5. Reductions of sTB

5.1 Reduction to Supersymmetric KdV

We have pointed out in sec. 2 that the Two Boson system reduces to various known integrable systems. For instance, the KdV equation can be embedded into the Two Boson system in a nonstandard Lax representation [9]. Let us first show that the sKdV [7,8] can, similarly, be embedded into the sTB [11].

Let us consider the Lax operator for the sTB system, L, given in (3.9). It is straightforward to show from its structure that

$$(L^3)_{\geq 1} = D^6 + 3D\Phi_1 D^2 - 3D^2(D\Phi_0)D^2 + 3(D\Phi_0)^2 D^2 + 6\Phi_1(D\Phi_0)D \tag{5.1}$$

It can now be shown in a simple manner that the nonstandard Lax equation,

$$\frac{\partial L}{\partial t} = [L, (L^3)_{\geq 1}] \tag{5.2}$$

leads to the dynamical equations

$$\frac{\partial \Phi_1}{\partial t} = -(D^6\Phi_1) - 3D^2\Big(\Phi_1(D\Phi_0)^2 + (D^2\Phi_1)(D\Phi_0) + \Phi_1(D\Phi_1)\Big) \tag{5.3a}$$

$$\frac{\partial \Phi_0}{\partial t} = -(D^6\Phi_0) + 3D\Big(\Phi_1(D^2\Phi_0) - 2(D\Phi_1)(D\Phi_0) - \frac{1}{3}(D\Phi_0)^3 + (D\Phi_0)(D^3\Phi_0)\Big) \tag{5.3b}$$

Given this, it is immediately clear that the identification

$$\begin{aligned} \Phi_0 &= 0 \\ \Phi_1 &= \Phi \end{aligned} \tag{5.4}$$

gives rise to the sKdV equation

$$\frac{\partial\Phi}{\partial t} = -(D^6\Phi) - 3D^2\left(\Phi(D\Phi)\right) \tag{5.5}$$

and that the sKdV equation is embedded in the sTB system as a nonstandard Lax equation (5.2) with the choice of the dynamical variables given in (5.4). We also see from (4.1) that with the condition (5.4), the even Hamiltonians vanish whereas the odd ones are the same as those for the sKdV system.

The reduction in (5.4) imposes a constraint on the sTB system and the Hamiltonian structures of the sKdV system can be obtained from those of the sTB system through a Dirac procedure [36]. If $\mathcal{D}$ represents any one of the Hamiltonian structures of the sTB system and H the appropriate odd Hamiltonian of the system, we have

$$\partial_t \begin{pmatrix} \Phi_0 \\ \\ \Phi_1 \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H}{\delta\Phi_0} \\ \\ \frac{\delta H}{\delta\Phi_1} \end{pmatrix} \tag{5.6}$$

Imposing the constraint (5.4) in (5.6), we obtain

$$\partial_t \begin{pmatrix} 0 \\ \\ \Phi \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{D}}_{11} & \overline{\mathcal{D}}_{12} \\ \overline{\mathcal{D}}_{21} & \overline{\mathcal{D}}_{22} \end{pmatrix} \begin{pmatrix} \overline{\frac{\delta H}{\delta\Phi_0}} = v_0 \\ \\ \overline{\frac{\delta H}{\delta\Phi_1}} = v_1 \end{pmatrix} \tag{5.7}$$

where $\overline{\mathcal{O}}$ denotes the quantities $\mathcal{O}$ calculated with the constraint (5.4). From (5.7), we note that consistency requires

$$v_0 = -\overline{\mathcal{D}}_{11}^{-1}\overline{\mathcal{D}}_{12}v_1 \tag{5.8}$$

and that we can write

$$\partial_t\Phi = \overline{\mathcal{D}}_{21}v_0 + \overline{\mathcal{D}}_{22}v_1 = \mathcal{D}^{\mathrm{sKdV}}v_1 \tag{5.9}$$

with (v_1 for odd Hamiltonians is the same as $\frac{\delta H}{\delta\Phi}$ for the sKdV system.)

$$\mathcal{D}^{\mathrm{sKdV}} = \overline{\mathcal{D}}_{22} - \overline{\mathcal{D}}_{21}\overline{\mathcal{D}}_{11}^{-1}\overline{\mathcal{D}}_{12} \tag{5.10}$$

We note now that if we use the second Hamiltonian structure (4.6), we obtain

$$(\overline{\mathcal{D}}_2)_{11}^{-1} = -\frac{1}{2}D(D^3+\Phi)^{-1}D \tag{5.11}$$

and (5.10) leads to the standard second Hamiltonian structure [8] of sKdV

$$\mathcal{D}_2^{\rm sKdV} = -\frac{1}{2}(D^5 + 3\Phi D^2 + (D\Phi)D + 2(D^2\Phi)) \tag{5.12}$$

To obtain the first Hamiltonian structure we should remember that when $\Phi_0 = 0$, all the even charges in (4.1) vanish. Consequently, the first Hamiltonian structure of the sKdV is obtained through Dirac reduction from $\mathcal{D}_0$ and not from $\mathcal{D}_1$, as would be naively expected. From the recursion relations, we know that

$$\mathcal{D}_0 = R^{-1}\mathcal{D}_1 \tag{5.13}$$

where R is the recursion operator of the sTB given in (4.9). It is easy to show that

$$\overline{R}^{-1} = \begin{pmatrix} 4D\left(\mathcal{D}_2^{\rm sKdV}\right)^{-1}D & -2D\left(\mathcal{D}_2^{\rm sKdV}\right)^{-1}D^3 \\ & \\ 2D^3\left(\mathcal{D}_2^{\rm sKdV}\right)^{-1}D & 2D^2(D^3+\Phi)^{-1}D^{-1}(\Phi D^2 + D^2\Phi)\left(\mathcal{D}_2^{\rm sKdV}\right)^{-1}D^3 \end{pmatrix} \tag{5.14}$$

Using the Dirac reduction relation (5.10) for the Hamiltonian structure (5.13), we obtain

$$\mathcal{D}_1^{\rm sKdV} = (\overline{\mathcal{D}}_0)_{22} - (\overline{\mathcal{D}}_0)_{21}(\overline{\mathcal{D}}_0)_{11}^{-1}(\overline{\mathcal{D}}_0)_{12} \tag{5.15}$$

which takes the form

$$\mathcal{D}_1^{\rm sKdV} = -2D^2(D^3+\Phi)^{-1}D^2 \tag{5.16}$$

This is the nonlocal structure for the sKdV obtained in refs. [37-40], but our derivation [11] also shows that this structure satisfies Jacobi identity since the Hamiltonian structures of the sTB system do. (The proof of Jacobi identity for the structure in (5.16) is nontrivial and, to the best of our knowledge, has not been demonstrated directly.)

5.2 Reduction to Supersymmetric NLS

As we have pointed out, the field redefinition (2.25) takes the two boson equation to the nonlinear Schrödinger equation. Therefore, it would seen natural to find a field redefinition that will take the sTB equation to the sNLS equation. In fact with the field redefinitions [10] (the only consistent ones and which reduces to (2.25) in the bosonic limit)

$$\begin{aligned} \Phi_0 &= -\left(D\ln(DQ)\right) + \left(D^{-1}(\overline{Q}Q)\right) \\ \Phi_1 &= -\overline{Q}(DQ) \end{aligned} \tag{5.17}$$

in terms of fermionic superfields

$$
\begin{aligned}
Q &= \psi + \theta q \\
\overline{Q} &= \overline{\psi} + \theta \overline{q}
\end{aligned}
\tag{5.18}
$$

which are complex conjugate of each other, equations (3.10), after a slightly involved derivation, reduce to

$$
\begin{aligned}
\frac{\partial Q}{\partial t} &= -(D^4 Q) + 2\left(D((DQ)\overline{Q})\right) Q \\
\frac{\partial \overline{Q}}{\partial t} &= (D^4 \overline{Q}) - 2\left(D((D\overline{Q})Q)\right) \overline{Q}
\end{aligned}
\tag{5.19}
$$

This is the supersymmetric nonlinear Schrödinger equation that is integrable. In fact, the supersymmetrization of the NLS equation [41,42] leads to the following equations

$$
\begin{aligned}
\frac{\partial Q}{\partial t} &= -(D^4 Q) + 2\alpha(D\overline{Q})(DQ)Q - 2\gamma\overline{Q}Q(D^2 Q) + 2(1-\alpha)\overline{Q}(DQ)^2 \\
\frac{\partial \overline{Q}}{\partial t} &= (D^4 \overline{Q}) - 2\alpha(DQ)(D\overline{Q})\overline{Q} + 2\gamma Q\overline{Q}(D^2 \overline{Q}) - 2(1-\alpha)Q(D\overline{Q})^2
\end{aligned}
\tag{5.20}
$$

However, it has been shown in [42] that various tests of integrability hold for the system of equations (5.20) only for

$$
\alpha = -\gamma = 1 \tag{5.21}
$$

suggesting that the system is integrable only for these values of the parameters. Equation (5.19) is indeed the equation (5.20) for the values of the parameter in (5.21) and we have obtained it from the integrable sTB equation through a field redefinition and, therefore, it is integrable.

We can also obtain a scalar Lax operator for the sNLS from the Lax operator for the sTB system [12]. Using the field identifications (5.17) in (3.9) we get

$$
\begin{aligned}
L =& D^2 + \frac{(D^3 Q)}{(DQ)} - \overline{Q}Q - D^{-1}\overline{Q}(DQ) \\
=& G\widetilde{L}G^{-1}
\end{aligned}
\tag{5.22}
$$

where

$$
\begin{aligned}
G =& (DQ)^{-1} \\
\widetilde{L} =& D^2 - \overline{Q}Q - (DQ)D^{-1}\overline{Q}
\end{aligned}
\tag{5.23}
$$

We see that the two Lax operators, L and $\widetilde{L}$, are related by a gauge transformation in the superspace. Although, this resembles the bosonic case (see eqs. (2.27) and (2.28)), $\widetilde{L}$ does

not lead to any consistent equation in the standard or nonstandard representation of the Lax equation. However, the formal adjoint of $\widetilde{L}$

$$\mathcal{L} = \widetilde{L}^* = -\left(D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)\right) \tag{5.24}$$

gives the sNLS equations (5.19) via the nonstandard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\mathcal{L}, \left(\mathcal{L}^2\right)_{\geq 1}\right] \tag{5.25}$$

This differs from the bosonic case (as we had pointed out earlier), since it is only the formal adjoint of the gauge transformed Lax operator which leads to the consistent equations. The other difference from the bosonic case is the fact that the supersymmetric generalization of NLS is obtained as a nonstandard Lax equation in superspace.

The relations in (5.17) are invertible and can be formally written as

$$Q = \left(D^{-1}\mathrm{e}^{\left(D^{-2}\left(-(D\Phi_0)+\Phi_1(L^{-1}\Phi_0)\right)\right)}\right) \tag{5.26a}$$

$$\overline{Q} = -\,\Phi_1 \mathrm{e}^{\left(-D^{-2}\left(-(D\Phi_0)+\Phi_1(L^{-1}\Phi_0)\right)\right)} \tag{5.26b}$$

and these ((5.17) and (5.26)) define the connecting relations between the sNLS and the sTB equations.

The conserved charges for the sNLS equation can be obtained from

$$H_n = \frac{1}{n}\mathrm{sTr}\,\mathcal{L}^n \tag{5.27}$$

The first few conserved charges have the form

$$\begin{aligned}
H_1 &= \int dz\,(DQ)\overline{Q} \\
H_2 &= \int dz\,(D^3Q)\overline{Q} \\
H_3 &= \int dz\,\left[(D^2\overline{Q})(DQ)\overline{Q}Q - (D^3Q)(D^2\overline{Q}) - (D\overline{Q})(DQ)^2\overline{Q}\right]
\end{aligned} \tag{5.28}$$

Since the sTB and the sNLS are related through a field redefinition and we already know the Hamiltonian structures of the sTB system, we can also obtain the Hamiltonian

structures of the sNLS equations [11] in the following way. Let us define the transformation matrix between the variables of the two systems as

$$P = \left[\frac{\delta(\Phi_0, \Phi_1)}{\delta(Q, \overline{Q})}\right] \tag{5.29}$$

where $\left[\frac{\delta(\Phi_0,\Phi_1)}{\delta(Q,\overline{Q})}\right]$ is the matrix formed from the Fréchet derivatives of Φ_0 and Φ_1 with respect to Q and $\overline{Q}$. Then, the Hamiltonian structures of the two systems are related by [43]

$$\mathcal{D} = P\,\mathcal{D}^{\mathrm{sNLS}}P^* \tag{5.30}$$

where P^* is the formal adjoint of P (with the matrix transposed). Explicitly, we have

$$P = \begin{pmatrix} -D(DQ)^{-1}D + D^{-1}\overline{Q} & -D^{-1}Q \\ -\overline{Q}D & -(DQ) \end{pmatrix} \tag{5.31}$$

It is easy to see that the matrix P factorizes in the following form

$$P = \widetilde{P}G \tag{5.32}$$

where

$$\widetilde{P} = \begin{pmatrix} -D^{-1} & -D^{-1}(DQ)^{-1}QD^2 \\ 0 & -D^2 \end{pmatrix} \tag{5.33a}$$

$$G = \begin{pmatrix} -\mathcal{L}(DQ)^{-1}D & 0 \\ D^{-2}\overline{Q}D & D^{-2}(DQ) \end{pmatrix} \tag{5.33b}$$

This may suggest that there is another equation to which both the sTB and the sNLS equations can be transformed [23]. But we will not go into a discussion of this.

From (5.30) we note that the Hamiltonian structures of the sNLS system can be obtained from those of the sTB system (written in terms of Q and $\overline{Q}$) as

$$\mathcal{D}^{\mathrm{sNLS}} = P^{-1}\mathcal{D}\,(P^*)^{-1} = G^{-1}\widetilde{P}^{-1}\mathcal{D}\,(\widetilde{P}^*)^{-1}G^{-1} \tag{5.34}$$

where the inverse matrix has the form

$$\begin{aligned} P^{-1} &= \begin{pmatrix} D^{-1}(DQ)\mathcal{L}^{-1}D & -D^{-1}(DQ)\mathcal{L}^{-1}Q(DQ)^{-1} \\ -\overline{Q}\mathcal{L}^{-1}D & -(1-\overline{Q}\mathcal{L}^{-1}Q)(DQ)^{-1} \end{pmatrix} \\ &= G^{-1}\widetilde{P}^{-1} \end{aligned} \tag{5.35}$$

For completeness, we record here the form of the inverse matrices

$$\widetilde{P}^{-1} = \begin{pmatrix} -D & (DQ)^{-1}Q \\ 0 & -D^2 \end{pmatrix} \tag{5.36a}$$

$$G^{-1} = \begin{pmatrix} -D^{-1}(DQ)\mathcal{L}^{-1} & 0 \\ \overline{Q}\mathcal{L}^{-1} & (DQ)^{-1}D^2 \end{pmatrix} \tag{5.36b}$$

Now, using the second Hamiltonian structure (4.6) of the sTB system, we obtain from (5.34)

$$\mathcal{D}_2^{\mathrm{sNLS}} = P^{-1}\mathcal{D}_2(P^*)^{-1} \tag{5.37}$$

which alongwith (5.35) gives after some tedious algebra

$$\mathcal{D}_2^{\mathrm{sNLS}} = \begin{pmatrix} -QD^{-1}Q & -\frac{1}{2}D + QD^{-1}\overline{Q} \\ -\frac{1}{2}D + \overline{Q}D^{-1}Q & -\overline{Q}D^{-1}\overline{Q} \end{pmatrix} \tag{5.38}$$

This is second Hamiltonian structure that was derived in ref. [42]. We note that it is enormously simpler to check this result by noting that $\mathcal{D}_2$ factorizes as

$$\mathcal{D}_2 = P\,\mathcal{D}_2^{\mathrm{sNLS}}P^* \tag{5.39}$$

with $\mathcal{D}_2^{\mathrm{sNLS}}$ given by (5.38).

The first Hamiltonian structure for the sNLS [11] can also be derived from (5.34) as

$$\mathcal{D}_1^{\mathrm{sNLS}} = P^{-1}\mathcal{D}_1(P^*)^{-1} \tag{5.40}$$

which upon using equations (4.3) and (5.35), gives

$$\mathcal{D}_1^{\mathrm{sNLS}} = \begin{pmatrix} -D^{-1}(DQ)\Delta(DQ)D^{-1} & \begin{matrix} D^{-1}(DQ)\Delta\overline{Q} \\ +D^{-1}(DQ)\mathcal{L}^{-1}D^2(DQ)^{-1} \end{matrix} \\ \begin{matrix} \overline{Q}\Delta(DQ)D^{-1} \\ +(DQ)^{-1}D^2(\mathcal{L}^*)^{-1}(DQ)D^{-1} \end{matrix} & \begin{matrix} -\overline{Q}\mathcal{L}^{-1}D^2(DQ)^{-1} - (DQ)^{-1}D^2(\mathcal{L}^*)^{-1}\overline{Q} \\ -\overline{Q}\Delta\overline{Q} \end{matrix} \end{pmatrix} \tag{5.41}$$

where we have defined

$$\Delta \equiv \mathcal{L}^{-1}\left(D^2((DQ)^{-1}Q)\right)(\mathcal{L}^*)^{-1} \tag{5.42}$$

Like the first Hamiltonian structure (5.16) of the sKdV equation, we note that this structure is highly nonlocal.

It can now be checked explicitly that the sNLS equations (5.19) can be written in the Hamiltonian form

$$\partial_t \begin{pmatrix} Q \\ \\ \overline{Q} \end{pmatrix} = \mathcal{D}_1^{\mathrm{sNLS}} \begin{pmatrix} \frac{\delta H_3}{\delta Q} \\ \\ \frac{\delta H_3}{\delta \overline{Q}} \end{pmatrix} = \mathcal{D}_2^{\mathrm{sNLS}} \begin{pmatrix} \frac{\delta H_2}{\delta Q} \\ \\ \frac{\delta H_2}{\delta \overline{Q}} \end{pmatrix} \tag{5.43}$$

where the Hamiltonians are defined as in (5.28). A direct check of (5.43) for the first Hamiltonian structure (5.41) is quite involved. But we note that it is much easier, instead, to check that

$$\begin{pmatrix} \frac{\delta H_3}{\delta Q} \\ \\ \frac{\delta H_3}{\delta \overline{Q}} \end{pmatrix} = \left(\mathcal{D}_1^{\mathrm{sNLS}}\right)^{-1} \begin{pmatrix} -(D^4 Q) + 2\left(D((DQ)\overline{Q})\right) Q \\ \\ (D^4 \overline{Q}) - 2\left(D((D\overline{Q})Q)\right) \overline{Q} \end{pmatrix} \tag{5.44}$$

The two Hamiltonian structures (5.38) and (5.41) automatically satisfy the Jacobi identity since the Hamiltonian structures of the sTB system do and define a recursion operator which would relate all the Hamiltonian structures as well as the conserved quantities of the system in a standard manner. However, as is clear from the structure of the first Hamiltonian structure, this recursion operator is extremely nontrivial and nonlocal.

5.3 Reduction to Supersymmetric mKdV

Finally, let us note that if we identify

$$\overline{Q} = Q \tag{5.45}$$

in (5.24), then, the nonstandard Lax equation

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\mathcal{L}, \left(\mathcal{L}^3\right)_{\geq 1}\right] \tag{5.46}$$

yields

$$\frac{\partial Q}{\partial t} = -(D^6 Q) + 3\left(D^2(Q(DQ))\right)(DQ) \tag{5.47}$$

This is the supersymmetric mKdV equation [8] and shows that we can embed the smKdV equation in the sNLS much like the sKdV can be embedded in the sTB equation [10].

However, as we have pointed out earlier, unlike the bosonic case, it is the adjoint operator in equation (5.24) which leads to the consistent dynamical equation. The Hamiltonian structures for smKdV can also be derived along the lines of our earlier discussions. We leave the derivation to the reader.

6. Nonstandard Supersymmetric KP Hierarchy

As we saw in sec. 5.2 the sNLS equation can be obtained from the nonstandard Lax equation (5.25) with the Lax operator (5.24). If we rewrite the operator $\mathcal{L}$ in (5.24) using the supersymmetric Liebnitz rule given in the Appendix we get

$$\begin{aligned} \mathcal{L} &= -\left(D^2 + \overline{Q}Q - \overline{Q}D^{-1}(DQ)\right) \\ &= -\left(D^2 + \sum_{n=-1}^{\infty} \Psi_n D^{-n}\right) \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \Psi_{-1} &= 0 \\ \Psi_n &= (-1)^{[\frac{n+1}{2}]}\, \overline{Q}(D^n Q), \quad n \geq 0 \end{aligned} \tag{6.2}$$

Ψ_{2n} (Ψ_{2n+1}) are bosonic (fermionic) superfields which define constrained variables much like the bosonic case (see (2.30) and (2.31)). The form of $\mathcal{L}$ in (6.1) resembles the Lax operator for the sKP hierarchy and, therefore, we can think of the sNLS system as a constrained sKP system of nonstandard kind (Viewed in this way the sTB system, given by the Lax operator (3.9), can also be thought of as a constrained, nonstandard sKP system.) However, we note that the Lax operator in the present case is an even parity operator [39,44,45] and not of the usual Manin-Radul form [7]. Our result also differs from the ones obtained in [14] through coset reduction.

This is a new system and, therefore, deserves further study [12]. Let us consider a general supersymmetric Lax operator of the form (6.1)

$$\begin{aligned} L &= D^2 + \Psi_0 + \Psi_1 D^{-1} + \Psi_2 D^{-2} + \cdots \\ &= D^2 + \sum_{n=0}^{\infty} \Psi_n D^{-n} \end{aligned} \tag{6.3}$$

where the superfields Ψ_n have the Grassmann parity

$$|\Psi_n| = \frac{1-(-1)^n}{2} \tag{6.4}$$

and have the form

$$\begin{aligned}\Psi_{2n} =& q_{2n} + \theta\phi_{2n} \\ \Psi_{2n+1} =& \phi_{2n+1} + \theta q_{2n+1}\end{aligned} \tag{6.5}$$

with the q_n (ϕ_n) being the bosonic (fermionic) components of the superfields.

The nonstandard flows associated with this sKP Lax operator are given by

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_{\geq 1}\, , L\right] \tag{6.6}$$

For $n = 1$, the flow is trivial and gives

$$\frac{\partial \Psi_n}{\partial t_1} = (D^2\Psi_n) = \left(\frac{\partial \Psi_n}{\partial x}\right) \tag{6.7}$$

implying that the time coordinate t_1 can be identified with x.

For $n = 2$, the flow in (6.6) gives

$$\begin{aligned}\frac{\partial \Psi_n}{\partial t_2} =& (D^4\Psi_n) + 2(D^2\Psi_{n+2}) + 2\Psi_0(D^2\Psi_n) + 2\Psi_1(D\Psi_n) - 2(1+(-1)^n)\Psi_1\Psi_{n+1} \\ &+ 2\sum_{\ell\geq 1}\left\{-(-1)^{[\ell/2]}\begin{bmatrix} n+1 \\ \ell \end{bmatrix}\Psi_{n-\ell+2}(D^\ell\Psi_0) + (-1)^{[\ell/2]+n}\begin{bmatrix} n \\ \ell \end{bmatrix}\Psi_{n-\ell+1}(D^\ell\Psi_1)\right\}\end{aligned} \tag{6.8}$$

where the super binomial coefficients $\begin{bmatrix} n \\ \ell \end{bmatrix}$ are defined in the Appendix. Equations for the bosonic components, q's, can be obtained from (6.8) [12]. In the bosonic limit – when all the ϕ_n's are zero – if we set

$$q_{2n} = 0 \quad \text{and} \quad q_{2n+1} = u_n\,, \quad \text{for all } n \tag{6.9}$$

we end up with

$$\begin{aligned}\frac{\partial u_0}{\partial t_2} =& u_0'' + 2u_1' \\ \frac{\partial u_1}{\partial t_2} =& u_1'' + 2u_2' + 2u_0u_0' \\ \frac{\partial u_2}{\partial t_2} =& u_2'' + 2u_3' - 2u_0u_0'' + 4u_1u_0' \\ &\vdots\end{aligned} \tag{6.10}$$

which are nothing other than the t_2-flows for the standard KP hierarchy [32,46].

On the other hand, in the bosonic limit, if we set

$$q_{2n+1} = 0 \quad \text{and} \quad q_{2n} = u_n\,, \quad \text{for all } n \tag{6.11}$$

then we obtain

$$\begin{aligned}
\frac{\partial u_0}{\partial t_2} =& u_0'' + 2u_1' + 2u_0 u_0' \\
\frac{\partial u_1}{\partial t_2} =& u_1'' + 2u_2' + 2u_0 u_1' + 2u_0' u_1 \\
\frac{\partial u_2}{\partial t_2} =& u_2'' + 2u_3' - 2u_1 u_0'' + 2u_0 u_2' + 4u_2 u_0' \\
& \vdots
\end{aligned} \tag{6.12}$$

which are nothing other than the t_2-flows associated with the standard mKP hierarchy [32,46].

For $n = 3$, equation (6.6) gives

$$\begin{aligned}
\frac{\partial \Psi_n}{\partial t_3} =& (D^6\Psi_n) + 3(D^4\Psi_{n+2}) + 3(D^2\Psi_{n+4}) + 3\Psi_0(D^4\Psi_n) + 6\Psi_0(D^2\Psi_{n+2}) \\
& + 3\Psi_1(D^3\Psi_n) + 3\Psi_1(D\Psi_{n+2}) - 3(-1)^n\Psi_1(D^2\Psi_{n+1}) \\
& - 3(1+(-1)^n)\Psi_1\Psi_{n+3} - 3(1+(-1)^n)((D^2\Psi_1) + \Psi_3 + 2\Psi_1\Psi_0)\Psi_{n+1} \\
& + 3((D^2\Psi_0) + \Psi_2 + \Psi_0^2)(D^2\Psi_n) + 3((D^2\Psi_1) + \Psi_3 + 2\Psi_1\Psi_0)(D\Psi_n) \\
& + 3\sum_{\ell\geq 1}\Bigg\{ -(-1)^{[\ell/2]}\begin{bmatrix} n+3 \\ \ell \end{bmatrix}\Psi_{n-\ell+4}(D^\ell\Psi_0) \\
& \qquad + (-1)^{[\ell/2]+n}\begin{bmatrix} n+2 \\ \ell \end{bmatrix}\Psi_{n-\ell+3}(D^\ell\Psi_1) \\
& \qquad + (-1)^{[\ell/2]}\begin{bmatrix} n+1 \\ \ell \end{bmatrix}\Psi_{n-\ell+2}(D^\ell\left((D^2\Psi_0) + \Psi_2 + \Psi_0^2\right)) \\
& \qquad + (-1)^{[\ell/2]+n}\begin{bmatrix} n \\ \ell \end{bmatrix}\Psi_{n-\ell+1}(D^\ell\left((D^2\Psi_1) + \Psi_3 + 2\Psi_1\Psi_0\right))\Bigg\}
\end{aligned} \tag{6.13}$$

Bosonic components equations for q's can again be obtained from (6.13) and in the bosonic limit, with the identifications in (6.9), we obtain

$$\begin{aligned}
\frac{\partial u_0}{\partial t_3} =& u_0''' + 3u_1'' + 3u_2' + 6u_0 u_0' \\
\frac{\partial u_1}{\partial t_3} =& u_1''' + 3u_2'' + 3u_3' + 6u_0 u_1' + 6u_0' u_1 \\
& \vdots
\end{aligned} \tag{6.14}$$

which are the t_3-flows for the standard KP hierarchy. On the other hand, the identifications in (6.11) leads to

$$
\begin{aligned}
\frac{\partial u_0}{\partial t_3} =& u_0''' + 3u_1'' + 3u_2' + 3u_0u_0'' + 3(u_0')^2 + 6(u_1u_0)' + 3u_0^2u_0' \\
&\vdots
\end{aligned} \tag{6.15}
$$

These are the t_3-flows for the standard mKP hierarchy. So, this new system, namely, the nonstandard sKP hierarchy contains both the standard KP and the standard mKP flows in its bosonic limit. The supersymmetry and the nonstandard nature of the equation has unified bosonic hierarchies into a more general one.

Since the nonstandard sKP equation of (6.6) reduces to the standard KP flows in the bosonic limit let us examine the nature of the sKP equation that it leads to [12]. If we put

$$
\Psi_{2n} = 0\,, \quad \text{for all } n \tag{6.16}
$$

then, the first two nontrivial equations following from (6.8) are

$$
\begin{aligned}
\frac{\partial \Psi_1}{\partial t_2} =& (D^4\Psi_1) + 2(D^2\Psi_3) \\
\frac{\partial \Psi_3}{\partial t_2} =& (D^4\Psi_3) + 2(D^2\Psi_5) - 2(D(\Psi_1\Psi_3)) + 2\Psi_1(D^3\Psi_1)
\end{aligned} \tag{6.17}
$$

and the first nontrivial equation following from (6.13) has the form

$$
\frac{\partial \Psi_1}{\partial t_3} = (D^6\Psi_1) + 3(D^4\Psi_3) + 3(D^2\Psi_5) + 3(D^2(\Psi_1(D\Psi_1))) \tag{6.18}
$$

From (6.17) and (6.18), we obtain, after the identifications $t_2 = y$, $t_3 = t$ and $\Psi_1 = \Psi = \phi + \theta u$, the equation

$$
D^2\left(\frac{\partial \Psi}{\partial t} - \frac{1}{4}(D^6\Psi) - \frac{3}{2}(D^2(\Psi(D\Psi))) - \frac{3}{2}(D(\Psi(\partial^{-1}\frac{\partial \Psi}{\partial y})))\right) = \frac{3}{4}\frac{\partial^2\Psi}{\partial y^2} \tag{6.19}
$$

This is a supersymmetric generalization of the KP equation which differs from the usual Manin-Radul equation [7,47] because of the presence of the nonlocal terms. In components this equation yields

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' + \frac{3}{2}\phi\phi'' - \frac{3}{2}\phi'(\partial^{-1}\frac{\partial \phi}{\partial y}) - \frac{3}{2}\phi\frac{\partial \phi}{\partial y}\right) =& \frac{3}{4}\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t} - \frac{1}{4}\phi''' - \frac{3}{2}(u\phi)' - \frac{3}{2}u(\partial^{-1}\frac{\partial \phi}{\partial y}) + \frac{3}{2}\phi(\partial^{-1}\frac{\partial u}{\partial y})\right) =& \frac{3}{4}\frac{\partial^2 \phi}{\partial y^2}
\end{aligned} \tag{6.20}
$$

which in the bosonic limit, $\phi = 0$, reduces to the KP equation. However, these equations are not invariant under $y \leftrightarrow -y$ unlike the Manin-Radul equations. We also note that when we restrict the variables u and ϕ to be independent of y, these equations reduce to the supersymmetric KdV equation [7,8]. Therefore, equation (6.19) represents a new supersymmetric generalization of the KP equation.

In the next section, we will show that the sTB hierarchy contains, in addition to the conserved local charges, fermionic nonlocal charges that are conserved and lead to higher symmetries.

7. Nonlocal Charges

Occurrence of nonlocal conserved charges in supersymmetric integrable models such as the sKdV equation was first recognized in [48]. In ref. [49], a systematic procedure for their construction was given within the framework of the Gelfand-Dikii formalism. It was shown that while the local charges for the sKdV can be obtained from odd powers of the square root of the Lax operator, $L^{\frac{2n-1}{2}}$, the nonlocal ones arise from odd powers of the quartic roots, $L^{\frac{2n-1}{4}}$. For the sTB, since the local charges Q_n's are related to integer powers of L (eq. (3.13)), we can expect conserved nonlocal charges F_n's from odd powers of the square root of L [13], that is,

$$F_{\frac{2n-1}{2}} = \mathrm{sTr}\, L^{\frac{2n-1}{2}} \qquad n = 1, 2, \ldots \tag{7.1}$$

The square root of the Lax operator, L, for sTB (given in (3.9)) has the form

$$L^{1/2} = D + a_0 + a_1 D^{-1} + a_2 D^{-2} + a_3 D^{-3} + a_4 D^{-4} + a_5 D^{-5} + \cdots \tag{7.2}$$

where

$$\begin{aligned} a_0 =& 2(D^{-2}\Phi_1) - \Phi_0 \\ a_1 =& -(D^{-1}\Phi_1) \\ a_2 =& (D^{-1}(\Phi_0\Phi_1)) - 2(D^{-2}((D\Phi_0)\Phi_1)) + \Phi_0(D^{-1}\Phi_1) - \Phi_1 \\ a_3 =& \frac{1}{2}(D^{-1}\Phi_1)^2 - (D\Phi_0)(D^{-1}\Phi_1) + (D^{-1}((D\Phi_0)\Phi_1)) + (D\Phi_1) \end{aligned} \tag{7.3}$$

$$\int dz\, a_5 = \int dz \left[\mathcal{O}(Da_1) - \Big(D^{-1}\big(2a_3(D^2 a_0) - a_1 a_3 + (Da_1)(D\mathcal{O})\big)\Big)\right]$$

and we have defined

$$\mathcal{O} \equiv (D^2 a_0) + (Da_1) - 2a_2 \tag{7.4}$$

with the grading of the coefficients given by

$$|a_n| = n + 1 \tag{7.5}$$

The first three nonlocal charges can be obtained [13] after some tedious, but straightforward, calculations to be

$$\begin{aligned}
F_{1/2} &= -\int dz\,(D^{-1}\Phi_1)\\
F_{3/2} &= -\int dz\left[\frac{3}{2}(D^{-1}\Phi_1)^2 - \Phi_0\Phi_1 - \Big(D^{-1}\big((D\Phi_0)\Phi_1\big)\Big)\right]\\
F_{5/2} &= -\int dz\left[\frac{1}{6}(D^{-1}\Phi_1)^3 - \big(5(D^{-2}\Phi_1)\Phi_1 - 2\Phi_0\Phi_1 - 3(D\Phi_1) - (D^{-1}\Phi_1)^2\big)(D\Phi_0)\right.\\
&\qquad\left. + \Big(D^{-1}\big((D\Phi_1)\Phi_1 + \Phi_1(D\Phi_0)^2 - (D\Phi_1)(D^2\Phi_0)\big)\Big)\right]
\end{aligned} \tag{7.6}$$

These charges have been explicitly checked to be conserved under the flow (3.10). (Conservation, of course, follows from the structure of the Lax equation (3.7). But an explicit check assures that the form of the charges given in (7.6) are indeed correct.)

Let us note that all the nonlocal charges in (7.6) are fermionic and $[F_{\frac{2n-1}{2}}] = \frac{2n-1}{2}$. Also, even though these charges are expressed as superspace integrals, they are not invariant under the supersymmetry transformations (3.12). This is because while the superspace integral of a local function of superfields is automatically supersymmetric, this is not necessarily true for nonlocal functions. However, this is not a matter of worry since even the supersymmetry charge, in these integrable models, is not supersymmetric.

We note that the nonlocal charges of the sTB hierarchy in (7.6) reduce to those of the sKdV hierarchy, up to normalizations [49], when we set $\Phi_0 = 0$. This is not surprising since we have already shown that the sKdV system is contained in the sTB system. However, unlike the sKdV system, the nonlocal charges (7.6) are not related recursively by either R or $R^\dagger$, given by (4.9) and (4.20) respectively. It is likely that these fermionic charges generate fermionic flows with distinct Hamiltonian structures of their own which in turn

can give a "recursion" operator connecting them. In [50] odd flows based on Jacobian sKP hierarchies were studied for the sKdV case, and maybe this result can be generalized to the sTB hierarchy if we use, instead, the nonstandard sKP hierarchy of sec. 6. We also note here that using the transformation (5.17) we can obtain the nonlocal charges for the sNLS equation which should coincide with the ones constructed in [41].

Let us note that the supersymmetry transformations (3.12) are generated by the conserved fermionic charge

$$Q = -\int dx\ (\psi_1 J_0 + \psi_0 J_1) \tag{7.7}$$

through the first Hamiltonian structure in (4.5) as

$$\begin{aligned}
\delta_Q J_0 =& \epsilon\{Q, J_0\}_1 = \epsilon\psi_0' \\
\delta_Q J_1 =& \epsilon\{Q, J_1\}_1 = \epsilon\psi_1' \\
\delta_Q \psi_0 =& \epsilon\{Q, \psi_0\}_1 = \epsilon J_0 \\
\delta_Q \psi_1 =& \epsilon\{Q, \psi_1\}_1 = \epsilon J_1
\end{aligned} \tag{7.8}$$

We can also show easily with eqs. (7.7) and (4.5) that

$$\{Q, Q\}_1 = -Q_2 \tag{7.9}$$

which implies that the supersymmetry charge is not supersymmetric – rather it satisfies a graded Lie algebra. As we have mentioned earlier, the nonlocal charges are also not invariant under supersymmetry transformations (3.12) or (7.8) and, in fact, we obtain

$$\begin{aligned}
\{Q, F_{1/2}\}_1 =& Q_1 \\
\{Q, F_{3/2}\}_1 =& \frac{1}{2}Q_2 \\
\{Q, F_{5/2}\}_1 =& \frac{1}{3}Q_3 + \frac{1}{24}Q_1^3
\end{aligned} \tag{7.10}$$

Furthermore, since the supersymmetry charge Q as well as the nonlocal charges $F_{\frac{2n-1}{2}}$ are conserved, they are in involution with all the local conserved charges Q_n, i.e,

$$\begin{aligned}
\{Q_n, Q_m\}_1 =& 0 \\
\{Q_n, F_{\frac{2m-1}{2}}\}_1 =& 0 \\
\{Q_n, Q\}_1 =& 0
\end{aligned} \tag{7.11}$$

The algebra of the nonlocal charges is quite interesting as well and we list here only the first few relations

$$\begin{aligned}
\{F_{1/2},F_{1/2}\}_1 &= 0 & \{F_{3/2},F_{3/2}\}_1 &= 2Q_2 \\
\{F_{1/2},F_{3/2}\}_1 &= Q_1 & \{F_{3/2},F_{5/2}\}_1 &= \frac{7}{3}Q_3 + \frac{7}{24}Q_1^3 \\
\{F_{1/2},F_{5/2}\}_1 &= Q_2 & \{F_{5/2},F_{5/2}\}_1 &= 3Q_4 - \frac{5}{8}Q_1^2 Q_2
\end{aligned} \tag{7.12}$$

This shows that the algebra of the conserved charges Q, Q_n and $F_{\frac{2n-1}{2}}$, at least at these orders, closes with respect to the first Hamiltonian structure in (4.5). However, the algebra is not a linear Lie algebra, but it is a graded nonlinear algebra where the nonlinearity manifests in a cubic term. The canonical dimensions of the charges allows for higher order nonlinearity to be present even in the algebraic relations of these lower order charges, but the algebra appears to involve only a cubic nonlinearity. The Jacobi identity is seen to be trivially satisfied for this algebra since the Q_n's are in involution with all the fermionic charges (Q_n's are the Hamiltonians for the system and the fermionic charges are conserved.). We note here that the cubic terms in (7.10) and (7.12) arise from boundary contributions when nonlocal terms are involved. This can be illustrated using the following realization of the inverse operator

$$\left(\partial^{-1} f(x)\right) = \frac{1}{2}\int dy\, \epsilon(x-y) f(y)\,, \qquad \epsilon(x) = \begin{cases} -1, & x<0 \\ \ \ 0, & x=0 \\ +1, & x>0 \end{cases} \tag{7.13}$$

This will lead, for instance, to boundary terms of the type

$$\int dz\, (D^{-1}\Phi_1)^2 \Phi_1 = \frac{1}{3}\int dz\, D(D^{-1}\Phi_1)^3 = -\frac{1}{12}Q_1^3 \tag{7.14}$$

and this is the origin of the nonlinear terms.

Since the sTB hierarchy has three distinct Hamiltonian structures, one can ask other interesting questions such as what transformations does Q generate with the second structure or what is the algebra of the charges with respect to the second structure and so on. In fact, it has been shown in [13] that the charges do satisfy a graded algebra with respect to the second structure as well and the algebra continues to be a cubic algebra. In fact, the graded algebra obtained with the second structure appears to be sort of a shifted version of the previous one.

From (7.12) we see that the nonlocal charge $F_{3/2}$ has the same Poisson bracket with itself (except for normalizations) as the supersymmetry charge (see (7.9)). Thus, we can think of $F_{3/2}$ as also generating a set of supersymmetry transformations given by

$$\begin{aligned}
\delta_{3/2}J_0 =&\epsilon\{F_{3/2},J_0\}_1 = \epsilon(\psi_0' - 3\psi_1)\\
\delta_{3/2}J_1 =&\epsilon\{F_{3/2},J_1\}_1 = -2\epsilon\psi_1'\\
\delta_{3/2}\psi_0 =&\epsilon\{F_{3/2},\psi_0\}_1 = \epsilon J_1\\
\delta_{3/2}\psi_1 =&\epsilon\{F_{3/2},\psi_1\}_1 = \epsilon\left(3(\partial^{-1}J_1) - 2J_0\right)
\end{aligned}\tag{7.15}$$

These are distinct from the transformations in (3.12) and are nonlocal. Moreover, $F_{3/2}$ can be seen from (7.12) and (7.10) to have the same algebra, with the other generators of the algebra, as the supersymmetry charge, Q. This suggests that even though the sTB equation we have constructed, in section 3, appears to have an $N=1$ supersymmetry, in reality, it has an $N=2$ extended supersymmetry [15,16]. We will return to this question in some more detail in the sec. 8.

For the present, let us note that the graded algebra of our system appears to be a cubic algebra. Cubic algebras have been found earlier in studies of other systems such as the Heisenberg spin chains and the nonlinear sigma models [51-54] and appear to be a common feature when nonlocal charges are involved. The nonlinearity in these algebras can, in principle, be high. However, the interesting thing about these algebras is that it is possible to redefine the generators in a highly nonlinear and nontrivial way such that the algebra becomes a cubic algebra. This is quite well known in the case of the nonlinear sigma model [54] and we will describe below how this is achieved in the case of supersymmetric integrable systems through the example of sKdV (because it is a simpler system). We note here that cubic terms are also present in the algebra of nonlocal charges, in the case of the sKdV system [13], even though it has not been observed before. If we take the sKdV equation, $\Phi_t = -(D^6\Phi) + 3D^2(\Phi(D\Phi))$ and use the nonlocal charges, the Lax operator L as well as the second Hamiltonian structure given in [49] (This differs from (5.12) by constant factors and field redefinitions.), and take the conserved local charges of sKdV as

$$H_{2n-1} = \frac{2^{2n-1}}{2n-1}\mathrm{Tr}\, L^{\frac{2n-1}{2}}\tag{7.16}$$

the following algebra can be obtained in a straightforward manner

$$\begin{aligned}
\{J_{1/2},J_{1/2}\} =& -H_1 & \{J_{3/2},J_{3/2}\} =& 4H_3-\frac{1}{3}H_1^3 \\
\{J_{1/2},J_{3/2}\} =& 0 & \{J_{3/2},J_{5/2}\} =& 0 \\
\{J_{1/2},J_{5/2}\} =& -6H_3-\frac{1}{4}H_1^3 & \{J_{5/2},J_{5/2}\} =& -36H_5-\frac{9}{80}H_1^5
\end{aligned} \tag{7.17}$$

The important point to note is the appearance of the quintic term in the bracket of $J_{5/2}$ with itself. However, we can redefine

$$\begin{aligned}
\hat{J}_{1/2} =& J_{1/2} \\
\hat{J}_{3/2} =& J_{3/2} \\
\hat{J}_{5/2} =& J_{5/2}+\alpha H_1^2 J_{1/2}
\end{aligned} \tag{7.18}$$

where α can be chosen such that the algebra becomes cubic. From this we strongly believe that one can redefine the charges even in the case of sTB such that the right hand side of the algebra in (7.10) and (7.12) will have the closed form structure

$$a\,\hat{Q}_n+b\sum_{p+q+\ell=n}\hat{Q}_p\hat{Q}_q\hat{Q}_\ell \tag{7.19}$$

where a and b are numerical factors and n is the sum of the canonical dimensions of the left hand side of the algebra.

We conclude this section by noting that cubic algebras of this sort can be related to Yangians [51-55]. So, it is natural to expect that the algebra, in the present systems (both sKdV and sTB) also corresponds to a Yangian. There is, however, a difficulty with this. Namely, a Yangian starts out with a non-Abelian Lie algebra for the local charges. Here, in contrast, the algebra of the local charges, Q_n's, is involutive (Abelian). There may still be an underlying Yangian structure in this algebra and this remains an open question.

8. Conclusions

We have discussed in this paper the supersymmetric generalization of the two boson system and its various properties [10]. The supersymmetric system, much like its bosonic counterpart, appears to have a rich structure. It reduces to various other known supersymmetric integrable systems. It has a bi-Hamiltonian structure and since it reduces to

various other supersymmetric systems under field redefinitions or reductions, allows us to construct various quantities associated with these systems in a simple way. For example, as we have pointed out, the Hamiltonian structures associated with various systems can be obtained in a natural way [11]. The scalar Lax operator associated with the sNLS system follows quite simply and shows that it can be thought of as a constrained, nonstandard sKP hierarchy [12]. The constrained, nonstandard sKP hierarchy is in itself quite interesting because in addition to giving a new integrable supersymmetrization of the KP equation, it unifies all the KP and mKP flows into a single equation. We have also derived the nonlocal charges associated with the sTB system and their algebra appears to be a cubic algebra [13]. Most important, however, is the fact that even though we started with an $N = 1$ supersymmetric system, the system appears to have developed an $N = 2$ extended supersymmetry. We now return to a discussion of this and other recent results associated with this system [14-17].

We begin by noting that recently an $N = 2$ supersymmetrization of the Two Boson system has been obtained in ref. [15] from a direct supersymmetrization of the second Hamiltonian structure (2.18). The set of equations that result from [15] is given by

$$\frac{\partial J}{\partial t} = [\,\widetilde{D}, \overline{\widetilde{D}}\,]J' + 4J'J \tag{8.1}$$

where J is the $N = 2$ bosonic superfield (We have made the identifications $S = J_0$ and $R = J_1$.)

$$J = \frac{1}{2}J_0 + \widetilde{\theta}\xi + \overline{\widetilde{\theta}}\,\overline{\xi} + \frac{1}{2}\widetilde{\theta}\,\overline{\widetilde{\theta}}\left(\frac{1}{2}J_0' - J_1\right) \tag{8.2}$$

and $\widetilde{D}$ and $\overline{\widetilde{D}}$ are covariant derivatives (We are using tilde variables so that comparison with our superspace results in sec. 3 becomes simpler.)

$$\begin{aligned}\widetilde{D} &= \frac{\partial}{\partial\widetilde{\theta}} - \frac{1}{2}\overline{\widetilde{\theta}}\frac{\partial}{\partial x}\\ \overline{\widetilde{D}} &= \frac{\partial}{\partial\overline{\widetilde{\theta}}} - \frac{1}{2}\widetilde{\theta}\frac{\partial}{\partial x}\end{aligned} \tag{8.3}$$

which satisfy

$$\{\widetilde{D}, \overline{\widetilde{D}}\} = -\frac{\partial}{\partial x}\,, \qquad \{\widetilde{D}, \widetilde{D}\} = \{\overline{\widetilde{D}}, \overline{\widetilde{D}}\} = 0 \tag{8.4}$$

In components, (8.1) takes the form

$$\begin{aligned}\frac{\partial J_0}{\partial t} &= -J_0'' + 2J_0'J_0 + 2J_1' \\ \frac{\partial J_1}{\partial t} &= J_1'' + 2(J_1J_0)' + 8(\xi\overline{\xi})' \\ \frac{\partial \overline{\xi}}{\partial t} &= \overline{\xi}'' + 2(\overline{\xi}J_0)' \\ \frac{\partial \xi}{\partial t} &= \xi'' + 2(\xi J_0)'\end{aligned} \tag{8.5}$$

It is now easy to see, with the linear identifications,

$$\begin{aligned}\overline{\xi} &= -\frac{1}{2}(\psi_0' - \psi_1) \\ \xi &= \frac{1}{2}\psi_1\end{aligned} \tag{8.6}$$

that the set of equations in (8.5) are nothing other than the sTB equation (3.11). As we have pointed out earlier, the $N = 2$ supersymmetry is already contained in our algebra (7.9), (7.10) and (7.12). To compare the two results, we note that the $N = 2$ transformations of ref. [15] can be read out from the form of the superfield (8.2) and have the following form in components

$$\begin{aligned}\delta J_0 &= 2\epsilon\xi & \overline{\delta} J_0 &= 2\epsilon\overline{\xi} \\ \delta J_1 &= 2\epsilon\xi' & \overline{\delta} J_1 &= 0 \\ \delta\overline{\xi} &= -\frac{1}{2}\epsilon(J_0' - J_1) & \overline{\delta}\,\overline{\xi} &= 0 \\ \delta\xi &= 0 & \overline{\delta}\xi &= -\frac{1}{2}\epsilon J_1\end{aligned} \tag{8.7}$$

Using the identifications (8.6) we can write them in terms of the variables in equation (3.11) as

$$\begin{aligned}\delta J_0 &= \epsilon\psi_1 & \overline{\delta} J_0 &= -\epsilon(\psi_0' - \psi_1) \\ \delta J_1 &= \epsilon\psi_1' & \overline{\delta} J_1 &= 0 \\ \delta\psi_0 &= \epsilon\left(J_0 - (\partial^{-1}J_1)\right) & \overline{\delta}\psi_0 &= -\epsilon(\partial^{-1}J_1) \\ \delta\psi_1 &= 0 & \overline{\delta}\psi_1 &= -\epsilon J_1\end{aligned} \tag{8.8}$$

From the supersymmetry transformations (7.8) and (7.15) we immediately see that

$$3\delta = \delta_Q - \delta_{3/2} \qquad \text{and} \qquad -3\overline{\delta} = 2\delta_Q + \delta_{3/2} \tag{8.9}$$

showing that the $N = 2$ supersymmetry of ref. [15] is the same $N = 2$ supersymmetry that is already present in our sTB system, but is manifest in the variables of ref. [15].

We can translate our Hamiltonian structures to the new variables in a simple way through the discussions of section 5. To obtain the second Hamiltonian structure, $\widetilde{\mathcal{D}}_2$, in terms of the variables in eq. (8.5) from the ones in eq. (4.7), we proceed as in sec. 5.2. Defining the matrix formed from the Fréchet derivatives of $J_0, J_1, \overline{\xi}, \xi$ with respect to J_0, J_1, ψ_0, ψ_1 as

$$P = \left[\frac{\delta(J_0, J_1, \overline{\xi}, \xi)}{\delta(J_0, J_1, \psi_0, \psi_1)}\right] \tag{8.10}$$

and relating the two Hamiltonian structures (see (5.30)) by

$$\widetilde{\mathcal{D}}_2 = P\,\mathcal{D}_2 P^* \tag{8.11}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\partial & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{8.12}$$

we obtain in a straightforward manner ($\mathcal{D}_2$ is given in (4.7).)

$$\widetilde{\mathcal{D}}_2 = \begin{pmatrix} 2\partial & \partial J_1 - \partial^2 & -\overline{\xi} & \xi \\ J_0\partial + \partial^2 & \partial J_1 + J_1\partial & \xi\partial & \partial\xi + \xi\partial \\ \overline{\xi} & \partial\overline{\xi} & 0 & -\frac{1}{4}(J_1 - \partial J_0 + \partial^2) \\ -\xi & \partial\xi + \xi\partial & -\frac{1}{4}(J_1 - \partial J_0 + \partial^2) & 0 \end{pmatrix} \tag{8.13}$$

Written in components, the only nonvanishing Poisson brackets are given by

$$
\begin{aligned}
\{J_0(x), J_0(y)\}_2 =& 2\delta'(x-y) \\
\{J_0(x), J_1(y)\}_2 =& (J_0\delta(x-y))' - \delta''(x-y) \\
\{J_0(x), \overline{\xi}(y)\}_2 =& -\overline{\xi}\delta(x-y) \\
\{J_0(x), \xi(y)\}_2 =& \xi\delta(x-y) \\
\{J_1(x), J_1(y)\}_2 =& J_1'\delta(x-y) + 2J_1\delta'(x-y) \\
\{J_1(x), \overline{\xi}(y)\}_2 =& \overline{\xi}\delta'(x-y) \\
\{J_1(x), \xi(y)\}_2 =& \xi'\delta(x-y) + 2\xi\delta'(x-y) \\
\{\overline{\xi}(x), \xi(y)\}_2 =& -\frac{1}{4}J_1\delta(x-y) + \frac{1}{4}(J_0\delta(x-y))' - \frac{1}{4}\delta''(x-y)
\end{aligned}
\tag{8.14}
$$

It is worth noting here that the algebra, in terms of the new variables, is local and that it is nothing other than the twisted $N=2$ superconformal algebra [31] whose bosonic limit is the Virasoro-Kac-Moody algebra (2.18).

We also note that the first Hamiltonian structure can similarly be transformed into the new variables using (8.11), (8.12) and (4.4) and takes the form

$$
\widetilde{\mathcal{D}}_1 = \begin{pmatrix} 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}\partial \\ 0 & 0 & -\frac{1}{4}\partial & 0 \end{pmatrix}
\tag{8.15}
$$

which gives the following nonvanishing Poisson brackets in components

$$
\begin{aligned}
\{J_0(x), J_1(y)\}_1 =& \delta'(x-y) \\
\{\xi(x), \xi(y)\}_1 =& \frac{1}{4}\delta'(x-y)
\end{aligned}
\tag{8.16}
$$

This shows that both the first and the second Hamiltonian structures for the system are local in the new variables and that the sTB system, in this sense, is truly a bi-Hamiltonian system.

To compare the relation between the $N=2$ superfield of ref. [15] and our superfields in eq. (3.3), we note the following. Given J in (8.2), we can define two chiral superfields

which with the identification in (8.6) take the following form.

$$
\begin{aligned}
2\widetilde{D}J =& \exp\left(\frac{1}{2}\widetilde{\theta}\,\overline{\widetilde{\theta}}\,\frac{\partial}{\partial x}\right)(\psi_1 - \overline{\widetilde{\theta}}J_1) = \exp\left(\frac{1}{2}\widetilde{\theta}\,\overline{\widetilde{\theta}}\,\frac{\partial}{\partial x}\right)j \\
2\overline{\widetilde{D}}J =& \exp\left(-\frac{1}{2}\widetilde{\theta}\,\overline{\widetilde{\theta}}\,\frac{\partial}{\partial x}\right)\Big[(\psi_1 - \psi_0') + \widetilde{\theta}(J_1 - J_0')\Big] = \exp\left(-\frac{1}{2}\widetilde{\theta}\,\overline{\widetilde{\theta}}\,\frac{\partial}{\partial x}\right)\overline{j}
\end{aligned}
\tag{8.17}
$$

It is also easy to see that we can define

$$
\begin{aligned}
\theta =& \frac{1}{2}(\widetilde{\theta} - \overline{\widetilde{\theta}}) \\
\overline{\theta} =& \frac{1}{2}(\widetilde{\theta} + \overline{\widetilde{\theta}})
\end{aligned}
\tag{8.18}
$$

so that we can write the covariant derivative of section 3 (see eq. (3.1)) as

$$
D = \widetilde{D} - \overline{\widetilde{D}} \tag{8.19}
$$

If we now set $\overline{\theta} = 0$, then it is easy to see from (8.17) that

$$
\begin{aligned}
j =& \Phi_1 \\
\overline{j} =& (\Phi_1 - \Phi_0')
\end{aligned}
\tag{8.20}
$$

In other words, the $N = 1$ superfields are chirally related to the $N = 2$ superfields under appropriate definition of coordinates.

To conclude, we note that there are still some very interesting questions, in connection with the sTB system, that need further investigation. Among other things, a derivation of the Hamiltonian structures from a generalization of the Gelfand-Dikii brackets for supersymmetric nonstandard system remains an open question. Similarly, further analysis of the structure of the algebra of nonlocal charges as well as its possible connection with (super) Yangian is quite interesting.

Acknowledgements

This work was supported in part by the U.S. Department of Energy Grant No. DE-FG-02-91ER40685. J.C.B. would like to thank CNPq, Brazil, for financial support.

Appendix

The supersymmetric Leibnitz [7] rule is given by

$$D^k A = \sum_{j=0}^{\infty} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^{|A|(k+j)} (D^j A) D^{k-j} \qquad (A.1)$$

where for $k \geq 0$

$$\begin{bmatrix} k \\ j \end{bmatrix} = \begin{cases} \begin{pmatrix} [k/2] \\ [j/2] \end{pmatrix} & \text{for } k \geq j \text{ and } (k,j) \neq (0,1) \bmod 2 \\ 0 & \text{otherwise} \end{cases} \qquad (A.2)$$

and for $k < 0$

$$\begin{bmatrix} k \\ j \end{bmatrix} = (-1)^{[j/2]} \begin{bmatrix} -k+j-1 \\ j \end{bmatrix} \qquad (A.3)$$

where $[k/2]$ denotes the integral part of $k/2$ and $|A|$ is the Grassmann parity of A, and $|A| = 0\,(1)$ for A even (odd).

The relation

$$(-1)^{|A|}\big(D^{-1}(AB)\big) = A(D^{-1}B) - \Big(D^{-1}\big((DA)(D^{-1}B)\big)\Big) \qquad (A.4)$$

is very useful and can be easily proved through the Leibnitz rule. Also, to perform integration by parts we need the generalized formula [35,49] which holds for local functions A and B,

$$\int dz\,(D^n A)B = (-1)^{n|A| + \frac{n(n+1)}{2}} \int dz\, A(D^n B), \quad \text{for all } n \qquad (A.5)$$

We note, however, that for nonlocal functions, the surface terms can not always be neglected. Throughout this paper the parenthesis limit the action of the inverse (integral) operators.

In the supersymmetric case the Poisson Brackets satisfy

$$\{A,B\} = -\,(-1)^{|A||B|}\{B,A\} \qquad (A.6a)$$

$$\{AB,C\} = (-1)^{|B||C|}\{A,C\}B + A\{B,C\} \qquad (A.6b)$$

$$\{A,BC\} = (-1)^{|A||B|}B\{A,C\} + \{A,B\}C \qquad (A.6c)$$

and the Jacobi identity is

$$(-1)^{|A||C|}\{\{A,B\},C\} + (-1)^{|B||C|}\{\{C,A\},B\} + (-1)^{|A||B|}\{\{B,C\},A\} = 0 \qquad (A.7)$$

References

1. L.D. Faddeev and L.A. Takhtajan, “Hamiltonian Methods in the Theory of Solitons” (Springer, Berlin, 1987).
2. A. Das, “Integrable Models” (World Scientific, Singapore, 1989).
3. L. A. Dickey, “Soliton Equations and Hamiltonian Systems” (World Scientific, Singapore, 1991).
4. D. J. Gross and A. A. Midgal, Phys. Rev. Lett. **64**, 127 (1990); D. J. Gross and A. A. Midgal, Nucl. Phys. **B340**, 333 (1990); E. Brézin and V. A. Kazakov, Phys. Lett. 236B, 144 (1990); M. Douglas and S. H. Shenker, Nucl. Phys. **B335**, 635 (1990); A. M. Polyakov in “Fields, Strings and Critical Phenomena”, Les Houches 1988, ed. E. Brézin and J. Zinn-Justin (North-Holland, Amsterdam, 1989); L. Alvarez-Gaumé, Helv. Phys. Acta **64**, 361 (1991); P. Ginsparg and G. Moore, “Lectures on 2D String Theory and 2D Gravity” (Cambridge, New York, 1993).
5. L. Alvarez-Gaumé and J. L. Manẽs, Mod. Phys. Lett. **A6**, 2039 (1991); L. Alvarez-Gaumé, H. Itoyama, J. Manẽs and A. Zadra, Int. J. Mod. Phys. **A7**, 5337 (1992).
6. S. Stanciu, “Supersymmetric Integrable Hierarchies and String Theory”, Bonn University preprint, BONN-IR-94-07 (see also hep-th/9407189); M. Becker, “Non-Perturbative Approach to 2D-Supergravity and Super-Virasoro Constraints”, CERN preprint, CERN-TH.7173/94 (see also hep-th/9403129); and references therein.
7. Y. I. Manin and A. O. Radul, Commun. Math. Phys. **98**, 65 (1985).
8. P. Mathieu, J. Math. Phys. **29**, 2499 (1988).
9. B.A. Kupershmidt, Commun. Math. Phys. **99**, 51 (1985).
10. J. C. Brunelli and A. Das, Phys. Lett. **B337**, 303 (1994).
11. J. C. Brunelli and A. Das, “Bi-Hamiltonian Structure of the Supersymmetric Nonlinear Schrödinger Equation”, University of Rochester preprint UR-1421 (1995) (also hep-th/9505041).
12. J. C. Brunelli and A. Das, “A Nonstandard Supersymmetric KP Hierarchy”, University of Rochester preprint UR-1367 (1994) (also hep-th/9408049), to appear in the Rev. Math. Phys..

13. J. C. Brunelli and A. Das, "Properties of Nonlocal Charges in the Supersymmetric Two Boson Hierarchy", University of Rochester preprint UR-1417 (1995) (also hep-th/9504030).
14. F. Toppan, Int. J. Mod. Phys. **A10**, 895 (1995).
15. S. Krivonos and A. Sorin, "The Minimal $N = 2$ Superextension of the NLS Equation", preprint JINR E2-95-172 (also hep-th/9504084).
16. S. Krivonos, A. Sorin and F. Toppan, "On the Super-NLS Equation and its Relation with $N = 2$ Super-KdV within Coset Approach", preprint JINR E2-95-185 (also hep-th/9504138).
17. Z. Popowicz, Phys. Lett. **A194**, 375 (1994).
18. L.J.F. Broer, Appl. Sci. Res. **31**, 377 (1975).
19. D.J. Kaup, Progr. Theor. Phys. **54**, 396 (1975).
20. H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, Nucl. Phys. **B402**, 85 (1993); H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, "On W_∞ Algebras, Gauge Equivalence of KP Hierarchies, Two-Boson Realizations and their KdV Reductions", in Lectures at the VII J. A. Swieca Summer School, São Paulo, Brazil, January 1993, eds. O. J. P. Éboli and V. O. Rivelles (World Scientific, Singapore, 1994); H. Aratyn, E. Nissimov and S. Pacheva, Phys. Lett. **B314**, 41 (1993).
21. L. Bonora and C.S. Xiong, Phys. Lett. **B285**, 191 (1992); L. Bonora and C.S. Xiong, Int. J. Mod. Phys. **A8**, 2973 (1993).
22. M. Freeman and P. West, Phys. Lett. **295B**, 59 (1992).
23. J. Schiff, "The Nonlinear Schrödinger Equation and Conserved Quantities in the Deformed Parafermion and SL(2,**R**)/U(1) Coset Models", Princeton preprint IASSNS-HEP-92/57 (1992) (also hep-th/9210029).
24. D. J. Benney, Stud. Appl. Math. **L11**, 45 (1973).
25. J. C. Brunelli, A. Das and W.-J. Huang, Mod. Phys. Lett. **9A**, 2147 (1994).
26. A. Das and W.-J. Huang, J. Math. Phys. **33**, 2487 (1992).
27. W. Oevel and W. Strampp, Commun. Math. Phys. **157**, 51 (1993).
28. A. Das and S. Roy, J. Math. Phys. **32**, 869 (1991).
29. W.J. Huang, J. Math. Phys. **35**, 993 (1994).

30. K. Bardakci and M. B. Halpern, Phys. Rev. **D3**, 2493 (1971).

31. E. Witten, Commun. Math. Phys. **117**, 353 (1988); **118**, 411 (1988); Nucl. Phys. **B340** 281 (1990); T. Eguchi and S. Yang, Mod. Phys. Lett. **A4**, 1693 (1990); R. Dijkgraaf, E. Verlinde and H. Verlinde, "Notes on Topological String Theory and 2-D Quantum Gravity", Lectures given at Spring School on Strings and Quantum Gravity, Trieste, Italy, April 24-May 2, 1990 and at Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France, May 28-June 1, 1990.

32. W. Oevel and C. Rogers, Rev. Math. Phys. **5**, 299 (1993).

33. E. Date, M. Kashiwara, M. Jimbo and T. Miwa, in "Nonlinear Integrable Systems-Classical Theory and Quantum Theory", ed. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).

34. P. J. Olver , "Applications of Lie Groups to Differential Equations", Graduate Texts in Mathematics, Vol. 107 (Springer, New York, 1986).

35. P. Mathieu, Lett. Math. Phys. **16**, 199 (1988).

36. W. Oevel and O. Ragnisco, Physica **A161**, 181 (1989).

37. W. Oevel and Z. Popowicz, Comm. Math. Phys. **139**, 441 (1991).

38. J.M. Figueroa-O'Farril, J. Mas and E. Ramos, Leuven preprint KUL-TF-91/19 (1991).

39. J. M. Figueroa-O'Farrill, J. Mas and E. Ramos, Rev. Math. Phys. **3**, 479 (1991).

40. J. Barcelos-Neto and A. Das, J. Math. Phys. **33**, 2743 (1992).

41. G.H.M. Roelofs and P.H.M. Kersten, J. Math. Phys. **33**, 2185 (1992).

42. J. C. Brunelli and A. Das, J. Math. Phys. **36**, 268 (1995).

43. G. Wilson, Phys. Lett. **A132**, 445 (1988).

44. F. Yu, Nucl. Phys. **B375**, 173 (1992).

45. J. Barcelos-Neto, S. Ghosh and S. Roy, J. Math. Phys. **36**, 258 (1995).

46. Y. Ohta, J. Satsuma, D. Takahashi and T. Tokihiro, Progr. Theor. Phys. Suppl. **94**, 210 (1988); K. Kiso, Progr. Theor. Phys. **83**, 1108 (1990).

47. J. Barcelos-Neto, A. Das, S. Panda and S. Roy, Phys. Lett. **B282**, 365 (1992).

48. P. H. M. Kersten, Phys. Lett. **A134**, 25 (1988).

49. P. Dargis and P. Mathieu, Phys. Lett. **A176**, 67 (1993).

50. E. Ramos, Mod. Phys. Lett. **A9**, 3235 (1994).

51. D. Bernard and A. LeClair, Commun. Math. Phys. **142**, 99 (1989); D. Bernard, "An Introduction to Yangian Symmetries", in Integrable Quantum Field Theories, ed. L. Bonora et al., NATO ASI Series B: Physics vol. 310 (Plenum Press, New York, 1993).
52. J. Barcelos-Neto, A. Das, J. Maharana, Z. Phys. **30C**, 401 (1986);
53. N. J. Mackay, Phys. Lett. **B281**, 90 (1992); erratum-ibid. **B308**, 444 (1993).
54. E. Abdalla, M. C. B. Abdalla, J. C. Brunelli and A. Zadra, Commun. Math. Phys. **166**, 379 (1994).
55. T. Curtright and C. Zachos, Nucl. Phys. **B402**, 604 (1993).